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On the degenerate Elzaki transform *

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Abstract Based on the Fourier transform, Elzaki in 2011 (Tarig M.Elzaki, The new integral transform Elzaki transform , Global Journal of Pure and Applied Mathematics, 7(1), 57–64 (2011)) defined the Elzaki transform. Motivated by the work Kim and Kim (Taekyun Kim and Dae San Kim, Degenerate Laplace transform and degenerate gamma function, Russian Journal of Mathematical Physics, 24(2), 241–248 (2017)) on the introduction of the degenerate Laplace transform, in this paper, we introduce the degenerate ELzaki transform and investigate some of its properties and relations. In particular we find the degenerate Elzaki transforms of the degenerate sine, the degenerate cosine, the degenerate hyperbolic sine and the degenerate hyperbolic cosine functions. Moreover, we investigate a scale preserving theorem for the degenerate Elzaki transform and establish that the degenerate Elzaki transform is a theoretical dual transform to the degenerate Laplace transform.

Key words degenerate Elzaki transform, Upadhyaya transform, degenerate Upadhyaya transform.

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1 Introduction

In mathematics, the various transforms are often applied as a valuable tool for transforming a differential equation from one form to another form to make it easy to handle and thus amenable to solution. The

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well known Laplace transform, propounded by Pierre Simon Laplace around 1780, is very widely used extensively for this purpose in almost all branches of mathematics, engineering, physical and social sciences and the like, just to mention a few. As most of us are very well aware that the Laplace transform converts a higher order differential equation into a polynomial form, with the incorporation of the initial values and the boundary conditions from the very beginning, which is far easier to solve than solving the original differential equation directly. The concept of Laplace transformation plays a vital role in diverse areas of science and technology such as electric analysis, communication engineering, control engineering, linear system, analysis, statistics, optics, quantum physics, solution of partial differential operation, data mining, machine learning, signal processing, integrated circuits, etc. The major advantage of Laplace transform is that, it is defined for both the stable and the unstable systems. Particularly, the Laplace transform is heavily used in signal processing to transform the signal in time domain into a signal in a complex frequency domain and it is used for the easy analysis of signals and systems.

In control systems, the Laplace transform is used to manage the commands, it directs or regulates the behavior of other devices or systems. The Laplace transform converts the governing differential equations of a system or its components into simple algebraic forms allowing the control engineers to describe the system, in particular, a closed loop system as a chain of connected functional blocks. Machine learning focuses on prediction, based on known properties learned from the training data, in which the Laplace transform is used to determine the prediction and to analyze the step of knowledge in databases. Also, the Laplace transformation helps us to find out the current or charge flowing through an electrical circuit and it provides us some criteria for the analyzing the circuits. It is used to manufacture the required ICs and chips for computer systems. Thus, in short, we can say that the Laplace transform plays a vital role in almost all the areas of science and technology.

As our technology is advancing day by day, in order to find either the exact or numerical solutions of the mathematical differential equations encountered in this process, various variants of the classical Laplace transforms are proposed by many mathematicians over the past about three decades, which have extensively been applied to a very diverse variety of phenomenon in natural, physical and applied sciences besides economic and social sciences. An exhaustive description of almost all these new variants of the classical Laplace transforms existing in the mathematics research literature today can be found in the seminal work of Upadhyaya [18] on Upadhyaya transform, which is the most general and the most robust variant of the classical Laplace transform as it unifies and generalizes almost all the variants of the classical Laplace transform that today exist in the mathematics literature, as far as known to us, right almost up to the year 2021 itself by the time of writing of this paper by us (see also, Upadhyaya [19]).

Among the many variants of the classical Laplace transform existing in the literature, a very effective method to get the solution of differential and integral equations and a linear system of differential and integral equations is the Elzaki transform method introduced by Elzaki in 2011 [3] which is based on the Fourier transform, and it can still serve as an auxiliary method of solving the problems of engineering and applied sciences in addition to the Laplace transform. The Elzaki transform may be used to solve intricate problems in almost all the areas of sciences and applied sciences without resorting to a new frequency domain [3–10, 13, 20].

In 2017 Kim and Kim [12] defined and studied some properties of the degenerate gamma function and they also introduced for the first time the concept of the degenerate Laplace transform and obtained some of its properties. This work of Kim and Kim [12] was extended by the third author in a series of papers [14–17]. Very recently Duran [1] has defined and studied the properties of the degenerate Sumudu transform. In this paper we introduce the degenerate Elzaki transform and study a number of its properties which also includes the relationship between the degenerate Laplace transformation and the degenerate Elzaki transformation, which we call the duality relation between the degenerate Laplace transform and the degenerate Elzaki transform. We also mention here that the third author has already introduced the concept of the more general and versatile transform - the degenerate Upadhyaya transform in [18], therefore, the degenerate Elzaki transform, which we introduce in this paper and the degenerate Sumudu transform introduced by Duran [1] are both *particular cases* of the already introduced degenerate Upadhyaya transform [18] (see also Upadhyaya et al. [19, Remark 3.2, p. 35]).

2 Preliminaries

In this section we present the definitions of the Elzaki transform, the degenerate exponential function, the degenerate sine and cosine functions, the degenerate hyperbolic sine and hyperbolic cosine functions and the degenerate Laplace transform.

Definition 2.1. The Elzaki transform [3, (2), p. 57]: For a function $f(t)$ of exponential order belonging to the set A defined by

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, \text{ if } t \in (-1)^j \times [0, \infty), j = 1, 2, \right\}$$

where the constant M is finite and k_1 and k_2 may be finite or infinite, the Elzaki transform of $f(t)$ is defined by

$$\mathcal{E}[f(t)] = s\int_0^\infty f(t)e^{-\frac{t}{s}}dt = T(s), t \geq 0,$$

which may also be rewritten as

$$\mathcal{E}[f(t)] = s^2 \int_0^\infty f(st)e^{-t}dt, t \geq 0.$$

Thus,

$$\mathcal{E}[f(t)] = s^2 \int_0^\infty f(st)e^{-t}dt = s\int_0^\infty f(t)e^{-\frac{t}{s}}dt = T(s), t \geq 0.$$

Definition 2.2. The degenerate exponential function [12, (1.3), p. 241]: The degenerate exponential function denoted by e_λ^t is a function of two variables $\lambda \in (0, \infty)$ and $t \in R$ defined by

$$e_\lambda^t = (1 + \lambda t)^{\frac{1}{\lambda}}.$$

It may be noted that,

$$\lim_{\lambda \to 0+} e_\lambda^t = \lim_{\lambda \to 0+} (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t.$$

Definition 2.3. The degenerate Euler formula [12, (1.9), p. 242]: The degenerate Euler formula is defined by the relation

$$e_\lambda^{it} = \lim_{\lambda \to 0+} (1 + \lambda t)^{\frac{i}{\lambda}} = \cos_\lambda(t) + i \sin_\lambda(t).$$

It can be noted that,

$$\lim_{\lambda \to 0+} e_\lambda^{it} = \lim_{\lambda \to 0+} (1 + \lambda t)^{\frac{i}{\lambda}} = e^{it} = \cos t + \sin t.$$

Definition 2.4. The degenerate cosine and the degenerate sine functions [12, (1.12), p. 242]: The degenerate cosine and the degenerate sine functions are defined by the relations

$$\cos_\lambda(t) = \frac{e_\lambda^{it} + e_\lambda^{-it}}{2} \quad \text{and} \quad \sin_\lambda(t) = \frac{e_\lambda^{it} - e_\lambda^{-it}}{2i}.$$

Here, it may be easily observed that

$$\lim_{\lambda \to 0+} \cos_\lambda(t) = \cos t \quad \text{and} \quad \lim_{\lambda \to 0+} \sin_\lambda(t) = \sin t.$$

Definition 2.5. The degenerate cosine and the degenerate sine functions [12, (3.10), (3.11) p. 245]: The degenerate hyperbolic cosine and the degenerate hyperboli sine functions are defined by the relations

$$\cosh_\lambda(at) = \frac{1}{2}\left((1 + \lambda t)^{\frac{a}{\lambda}} + (1 + \lambda t)^{-\frac{a}{\lambda}}\right), \quad \sinh_\lambda(at) = \frac{1}{2}\left((1 + \lambda t)^{\frac{a}{\lambda}} - (1 + \lambda t)^{-\frac{a}{\lambda}}\right).$$

Definition 2.6. The degenerate Laplace transform [12, (3.1) p. 244]: Let $\lambda \in (0, \infty)$ and let $f(t)$ be a function defined for $t \geq 0$, then the integral

$$\mathcal{L}_\lambda[f(t)] = \int_0^\infty (1 + \lambda t)^{\frac{-s}{\lambda}} f(t)dt,$$

is said to be the degenerate Laplace transform of f if the integral converges, it is also denoted by $\mathcal{L}_\lambda[f(t)] = F_\lambda(s)$.

3 The degenerate Elzaki transform

For $\lambda \in (0, \infty)$ and for a function $f(t)$ defined for $t \geq 0$ we define the *degenerate Elzaki transform* of $f(t)$ by the relation

$$\mathcal{E}_\lambda[f(t)] = T_\lambda(s) = s^2 \int_0^\infty (1+\lambda t)^{\frac{-1}{\lambda}} f(st)dt. \tag{3.1}$$

Let $w = st$, then the equation (3.1) becomes

$$\begin{aligned}\mathcal{E}_\lambda[f(t)] &= s^2 \int_0^\infty e_\lambda^{-t} f(st)dt \\ &= s^2 \int_0^\infty e_\lambda^{\frac{-w}{s}} f(w)\frac{dw}{s} \\ T_\lambda(s) = s \int_0^\infty e_\lambda^{\frac{-w}{s}} f(w)dw &= s\int_0^\infty e_\lambda^{\frac{-t}{s}} f(t)dt.\end{aligned}$$

Now we give below the duality relation between the degenerate Elzaki transform and the degenerate Laplace transform.

3.1 Duality relation between the degenerate Elzaki transform and the degenerate Laplace transform

Let $T_\lambda(s)$ be the degenerate Elzaki transform of $f(t)$ and $F_\lambda(s)$ be the degenerate Laplace transform of $f(t)$, i.e., $\mathcal{E}_\lambda(f(t)) = T_\lambda(s)$ and $\mathcal{L}_\lambda(f(t) = F_\lambda(s)$. Then from (3.1) follows that,

$$\mathcal{E}_\lambda[f(t)] = s\int_0^\infty (1+\lambda t)^{\frac{-1}{\lambda s}} f(t)dt = T_\lambda(s). \tag{3.2}$$

Now, from the Definition 2.6, we have

$$\mathcal{L}_\lambda[f(t)] = \int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)dt = F_\lambda(s).$$

This implies that

$$F_\lambda\left(\frac{1}{s}\right) = \int_0^\infty (1+\lambda t)^{\frac{-1}{\lambda s}} f(t)dt. \tag{3.3}$$

Using (3.3) in (3.2) we get

$$\mathcal{E}_\lambda[f(t)] = T_\lambda(s) = sF_\lambda\left(\frac{1}{s}\right). \tag{3.4}$$

Now, replace s by $\frac{1}{s}$ in (3.4) we have the following desired duality relation between the degenerate Elzaki transform and the degenerate Laplace transform

$$\begin{aligned}T_\lambda\left(\frac{1}{s}\right) &= \frac{1}{s}F_\lambda(s) \\ F_\lambda(s) &= sT_\lambda\left(\frac{1}{s}\right).\end{aligned} \tag{3.5}$$

4 Properties of the degenerate Elzaki transform

In this section we establish some properties of the degenerate Elzaki transform.

Theorem 4.1. *For any constant real numbers α and β we have the following linearity property of the degenerate Elzaki transform:*

$$\mathcal{E}_\lambda[\alpha f(t) + \beta g(t)] = \alpha\mathcal{E}_\lambda[f(t)] + \beta\mathcal{E}_\lambda[g(t)].$$

Proof. The proof is self evident. □

Theorem 4.2. *The degenerate Elzaki transforms of the degenerate cosine function and the degenerate sine function are given by*

$$\mathcal{E}_\lambda[\cos_\lambda(at)] = \frac{s^2(1-\lambda s)}{(1-\lambda s)^2 + (as)^2} \quad \text{and} \quad \mathcal{E}_\lambda[\sin_\lambda(at)] = \frac{s^3 a}{(1-\lambda s)^2 + (as)^2}.$$

Proof. We apply (3.4) to the following results proved by Kim and Kim [12, (3.8), p. 244 and (3.9) p. 245]

$$\mathcal{L}_\lambda[\cos_\lambda(at)] = \frac{s-\lambda}{(s-\lambda)^2+a^2} \quad \text{and} \quad \mathcal{L}_\lambda[\sin_\lambda(at)] = \frac{a}{(s-\lambda)^2+a^2},$$

by first taking $\mathcal{L}_\lambda[\cos_\lambda(at)] = F_\lambda(s)$ we have

$$\begin{aligned}\mathcal{E}_\lambda[\cos_\lambda(at)] &= s\frac{\frac{1}{s}-\lambda}{(\frac{1}{s}-\lambda)^2+a^2},\\ &= \frac{s^2(1-\lambda s)}{(1-\lambda s)^2+(as)^2},\end{aligned}$$

and similarly then by taking $\mathcal{L}_\lambda[\sin_\lambda(at)] = F_\lambda(s)$ we get

$$\begin{aligned}\mathcal{E}_\lambda[\sin_\lambda(at)] &= s\frac{a}{(\frac{1}{s}-\lambda)^2+(a)^2},\\ &= \frac{as^3}{(1-\lambda s)^2+(as)^2}.\end{aligned}$$

□

Theorem 4.3. *The degenerate Elzaki transforms of the degenerate hyperbolic functions are given by*

$$\mathcal{E}_\lambda[\cosh_\lambda(at)] = s^2\frac{1-\lambda s}{(1-\lambda s)^2-(as)^2} \quad \text{and} \quad \mathcal{E}_\lambda[\sinh_\lambda(at)] = \frac{as^3}{(1-\lambda s)^2-(as)^2}.$$

Proof. Using (3.4) in the following results proved by Kim and Kim [12, (3.14), (3.15) p. 245]

$$\mathcal{L}_\lambda[\cosh_\lambda(at)] = \frac{s-\lambda}{(s-\lambda)^2-a^2} \quad \text{and} \quad \mathcal{L}_\lambda[\sinh_\lambda(at)] = \frac{a}{(s-\lambda)^2-a^2},$$

by first taking $\mathcal{L}_\lambda[\cosh_\lambda(at)] = F_\lambda(s)$ we have

$$\begin{aligned}\mathcal{E}_\lambda[\cosh_\lambda(at)] &= s\frac{\frac{1}{s}-\lambda}{(\frac{1}{s}-\lambda)^2-a^2},\\ &= \frac{s^2(1-\lambda s)}{(1-\lambda s)^2-(as)^2},\end{aligned}$$

and similarly then by taking $\mathcal{L}_\lambda[\sinh_\lambda(at)] = F_\lambda(s)$ we get

$$\begin{aligned}\mathcal{E}_\lambda[\sinh_\lambda(at)] &= s\frac{a}{(\frac{1}{s}-\lambda)^2-(a)^2},\\ &= \frac{as^3}{(1-\lambda s)^2-(as)^2}.\end{aligned}$$

□

Theorem 4.4. *For $n \in \mathbb{N}$ and $s > (n+1)\lambda$,*

$$\mathcal{E}_\lambda(t^n) = \frac{n!s^{n+2}}{(1-\lambda s)(1-2\lambda s)\ldots(1-n\lambda s)(1-(n+1)\lambda s)}.$$

Proof. Using the following result of (3.4)

$$T_\lambda(s) = sF_\lambda\left(\frac{1}{s}\right)$$

in the following result proven by Kim and Kim [12, Theorem 3.2, p. 246] for $n \in \mathbb{N}$ and $s > (n+1)\lambda$

$$\mathcal{L}_\lambda(t^n) = \frac{n!}{(s-\lambda)(s-2\lambda)\ldots(s-n\lambda)(s-(n+1)\lambda)}$$

by taking in the present case $F_\lambda(s) = \mathcal{L}_\lambda(t^n)$ we get

$$\mathcal{E}_\lambda(t^n) = s\frac{n!}{(\frac{1}{s}-\lambda)(\frac{1}{s}-2\lambda)\dots(\frac{1}{s}-n\lambda)(\frac{1}{s}-(n+1)\lambda)}$$
$$= \frac{sn!}{\frac{1}{s^{n+1}}(1-\lambda s)(1-2\lambda s)\dots(1-n\lambda s)(1-(n+1)\lambda s)},$$

or,

$$\mathcal{E}_\lambda(t^n) = \frac{n!s^{n+2}}{(1-\lambda s)(1-2\lambda s)\dots(1-n\lambda s)(1-(n+1)\lambda s)}.$$

□

Theorem 4.5. *If $f, f^{(1)}, \dots, f^{(n-1)}$ are continuous on $(0,\infty)$ and are of degenerate exponential order and if $f^{(n)}(t)$ is piecewise continuous on $(0,\infty)$, then*

$$\mathcal{E}_\lambda[f^{(n)}(t)] = \frac{1}{s}\left(\frac{1}{s}+\lambda\right)\dots\left(\frac{1}{s}+(n-1)\lambda\right)\mathcal{E}_\lambda[(1+\lambda t)^{-n}f(t)]$$
$$- s\sum_{i=0}^{\infty} f^{(i)}(0)\left[\prod_{l=1}^{n-i-1}\left(\frac{1}{s}+(l-1)\lambda\right)\right],$$

where $f^{(n)}(t) = \left(\frac{d}{dt}\right)^n f(t)$.

Proof. We first prove this result by the method of induction. Applying directly the definition of the degenerate Elzaki transform to find the degenerate Elzaki transform of the function $f^{(1)}(t)$ we get

$$\mathcal{E}_\lambda\left(f^{(1)}(t)\right) = s\int_0^\infty (1+\lambda t)^{-\frac{1}{s\lambda}} f^{(1)}(t)\,dt,$$

which on integration by parts gives

$$\mathcal{E}_\lambda\left(f^{(1)}(t)\right) = s\left[(1+\lambda t)^{-\frac{1}{s\lambda}} f(t)\right]_{t=0}^{\infty} - s\times\frac{1}{s}\int_0^\infty (1+\lambda t)^{-\frac{1}{s\lambda}-1} f(t)\,dt.$$

A simple evaluation of the above expression, noting that $\lim_{t\to\infty}(1+\lambda t)^{-\frac{1}{s\lambda}}f(t) = 0$, gives

$$\mathcal{E}_\lambda\left(f^{(1)}(t)\right) = -sf(0) + \frac{1}{s}\mathcal{E}_\lambda\left((1+\lambda t)^{-1}f(t)\right).$$

If we let $g(t) = f^{(1)}(t)$ then

$$\mathcal{E}_\lambda\left(f^{(2)}(t)\right) = s\int_0^\infty (1+\lambda t)^{-\frac{1}{s\lambda}} f^{(2)}(t)\,dt = s\int_0^\infty (1+\lambda t)^{-\frac{1}{s\lambda}} g^{(1)}(t)\,dt = \mathcal{E}_\lambda\left(g^{(1)}(t)\right),$$

which on the application of the last result deduced for the degenerate Elzaki transform of the first derivative of a function yields,

$$\mathcal{E}_\lambda\left(f^{(2)}(t)\right) = -sg(0) + \frac{1}{s}\mathcal{E}_\lambda\left((1+\lambda t)^{-1}g(t)\right) = -sf^{(1)}(0) + \frac{1}{s}\mathcal{E}_\lambda\left((1+\lambda t)^{-1}f^{(1)}(t)\right)$$
$$= -sf^{(1)}(0) + \frac{1}{s}\times s\int_0^\infty (1+\lambda t)^{-\frac{1}{s\lambda}-1} f^{(1)}(t)\,dt.$$

Evaluating the second integral on the right hand side of the last equation by parts we can write that

$$\mathcal{E}_\lambda\left(f^{(2)}(t)\right) = -sf^{(1)}(0) + \left[(1+\lambda t)^{-\frac{1}{s\lambda}-1}f(t)\right]_{t=0}^{\infty} + \lambda\times\left(\frac{1}{\lambda s}+1\right)\int_0^\infty (1+\lambda t)^{-\frac{1}{s\lambda}-2} f(t)\,dt,$$

which, on noting that $\lim_{t\to\infty}(1+\lambda t)^{-\frac{1}{s\lambda}-1}f(t) = 0$, simplifies to

$$\mathcal{E}_\lambda\left(f^{(2)}(t)\right) = -sf^{(1)}(0) - f(0) + \frac{1}{s}\left(\frac{1}{s}+\lambda\right)\mathcal{E}_\lambda\left((1+\lambda t)^{-2}f(t)\right).$$

Proceeding ahead similarly after n-steps we can get the desired result. □

Alternative Proof of Theorem 4.3. We now prove Theorem 4.3 by making use of the duality relation (3.4) between the degenerate Elzaki transform and the degenerate Laplace transform. Kim and Kim [12, (3.20), p. 246] have shown that

$$\mathcal{L}_\lambda\left(f^{(1)}(t)\right)=-f(0)+s\mathcal{L}_\lambda\left((1+\lambda t)^{-1}f(t)\right)=-f(0)+s\int_0^\infty(1+\lambda t)^{-\frac{s}{\lambda}-1}f(t)\,dt=F_\lambda(s),\quad(\text{say}),$$

to which a simple application of (3.4) gives

$$\begin{aligned}\mathcal{E}_\lambda\left(f^{(1)}(t)\right)=sF_\lambda\left(\frac{1}{s}\right)&=s\left[-f(0)+\frac{1}{s}\int_0^\infty(1+\lambda t)^{-\frac{1}{s\lambda}-1}f(t)\,dt\right]\\&=-sf(0)+\frac{1}{s}\mathcal{E}_\lambda\left((1+\lambda t)^{-1}f(t)\right).\end{aligned}$$

Kim and Kim [12, (3.22), p. 246] have further shown that

$$\begin{aligned}\mathcal{L}_\lambda\left(f^{(2)}(t)\right)&=-f^{(1)}(0)-sf(0)+s(s+\lambda)\mathcal{L}_\lambda\left((1+\lambda t)^{-2}f(t)\right)\\&=-f^{(1)}(0)-sf(0)+s(s+\lambda)\int_0^\infty(1+\lambda t)^{-\frac{s}{\lambda}-2}f(t)\,dt=F_\lambda(s),\quad(\text{say}),\end{aligned}$$

to which an application of (3.4) yields that

$$\begin{aligned}\mathcal{E}_\lambda\left(f^{(2)}(t)\right)=sF_\lambda\left(\frac{1}{s}\right)&=s\left[-f^{(1)}(0)-\frac{1}{s}f(0)+\frac{1}{s}\left(\frac{1}{s}+\lambda\right)\int_0^\infty(1+\lambda t)^{-\frac{1}{s\lambda}-2}f(t)\,dt\right]\\&=-sf^{(1)}(0)-f(0)+\frac{1}{s}\left(\frac{1}{s}+\lambda\right)\mathcal{E}_\lambda\left((1+\lambda t)^{-2}f(t)\right).\end{aligned}$$

Finally, Kim and Kim [12, (3.23), p. 246] have shown that

$$\begin{aligned}\mathcal{L}_\lambda\left(f^{(n)}(t)\right)&=s(s+\lambda)\ldots(s+(n-1)\lambda)\mathcal{L}_\lambda\left((1+\lambda t)^{-n}f(t)\right)-\sum_{i=0}^{n-1}f^{(i)}(0)\left(\prod_{l=1}^{n-i-1}(s+(l-1)\lambda)\right)\\&=s(s+\lambda)\ldots(s+(n-1)\lambda)\int_0^\infty(1+\lambda t)^{-\frac{s}{\lambda}-n}f(t)\,dt\\&-\sum_{i=0}^{n-1}f^{(i)}(0)\left(\prod_{l=1}^{n-i-1}(s+(l-1)\lambda)\right)=F_\lambda(s),\quad(\text{say}),\end{aligned}$$

to which an application of (3.4) as illustrated above yields the desired result. □

Theorem 4.6. The first translation theorem for the degenerate Elzaki transform: *If $\mathcal{E}_\lambda(f(t))=T_\lambda(s)$ then*

$$\mathcal{E}_\lambda\left(e_\lambda^{at}f(t)\right)=(1-as)T_\lambda\left(\frac{s}{1-as}\right),\ \text{for } a\neq\frac{1}{s}.$$

Proof. From the definition of the degenerate Elzaki transform it follows that

$$\begin{aligned}\mathcal{E}_\lambda\left(e_\lambda^{at}f(t)\right)&=s\int_0^\infty(1+\lambda t)^{-\frac{1}{\lambda s}}e_\lambda^{at}f(t)dt=s\int_0^\infty(1+\lambda t)^{-\frac{1}{\lambda s}}(1+\lambda t)^{\frac{a}{\lambda}}f(t)dt\\&=(1-as)\cdot\left(\frac{s}{1-as}\right)\int_0^\infty(1+\lambda t)^{-\frac{(1-as)}{\lambda s}}f(t)dt=(1-as)T_\lambda\left(\frac{s}{1-as}\right),\end{aligned}$$

where, $a\neq\frac{1}{s}$. □

Theorem 4.7. The change of scale property for the degenerate Elzaki transform: *If $\mathcal{E}_\lambda(f(t))=T_\lambda(s)$ then*

$$\mathcal{E}_\lambda(f(at))=\frac{1}{a^2}T_{\frac{\lambda}{a}}(as),\ \text{for } a\neq 0.$$

Proof. Utilizing the definition of the degenerate Elzaki transform we see that

$$\mathcal{E}_\lambda\left(f(at)\right) = s\int_0^\infty (1+\lambda t)^{-\frac{1}{\lambda s}} f(at)dt,$$

in which the substitution $u = at$ with $du = adt$ leads us to

$$\begin{aligned}\mathcal{E}_\lambda\left(f(at)\right) &= \frac{s}{a}\int_0^\infty \left(1+\frac{\lambda}{a}u\right)^{-\frac{1}{\lambda s}} f(u)du = \frac{1}{a^2}\cdot(as)\int_0^\infty \left(1+\frac{\lambda}{a}u\right)^{-\frac{1}{\left(\frac{\lambda}{a}\right)\cdot as}} f(u)du \\ &= \frac{1}{a^2}T_{\frac{\lambda}{a}}\left(as\right), \quad a\neq 0.\end{aligned}$$

□

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