Comprehensive Decision-Making with Spherical Fermatean Neutrosophic Sets in Structural Engineering

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Abstract

This study introduces the Spherical Fermatean Neutrosophic Sets (SFNSs), representing a significant advancement in the realm of Neutrosophic Sets (NSs) and Fermatean neutrosophic sets (FNSs). In decision making scenarios involving diverse perspectives, a mere average of decision values may fail to capture the entire spectrum of viewpoints. To address this limitation, the SFNS is proposed as a comprehensive solution. It features a spherical representation that encompasses membership, non-membership and indeterminacy functions at its core, complemented by a defined radius. This spherical construct facilitates the encapsulation of all decision makers' opinions within its bounds, providing a holistic perspective. Leveraging its geometric structure, the SFNS excels in resolving ambiguity and risk with greater accuracy and effectiveness compared to conventional FNSs. This innovative approach aims to better accommodate the complexities of decision making involving diverse perspectives. Selecting the best material for a structural engineering project is given as numerical example at the end.

Keywords : Fermatean neutrosophic sets; Extension of Fermatean neutrosophic sets; Spherical Fermatean neutrosophic sets

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1 Introduction and Preliminaries

In the intricate domains of multi-criteria decision making (MCDM) and multi-group decision processes, decision theory is undergoing profound transformations. Individuals tasked with prioritizing, selecting, or allocating options amidst conflicting criteria must integrate quantitative metrics with qualitative insights. They navigate a complex interplay of data streams from diverse origins, often encountering imprecision, ambiguity and uncertainty due to subjective factors, incomplete information, inconsistent measurements and intricate interactions.

This study introduces the Spherical Fermatean Neutrosophic Set (SFNS) as a significant innovation within the domain of neutrosophic sets and Fermatean neutrosophic sets. Acknowledging the limitations of conventional decision making methods, especially in contexts characterized by diverse perspectives, the SFNS offers a comprehensive solution. Utilizing its spherical representation, the SFNS encapsulates membership, non-membership and indeterminacy functions, providing a holistic framework that better accommodates the complexities inherent in decision making processes.

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The objective of this study is to explore the SFNS as an innovative approach to decision making in scenarios involving diverse perspectives. By incorporating membership, non-membership and indeterminacy functions within a spherical construct, the SFNS captures the entire spectrum of viewpoints more effectively than conventional methods. Through empirical testing and analysis, the study demonstrates that SFNS offers greater accuracy and effectiveness in resolving ambiguity and risk. Ultimately, the SFNS enhances decision making processes by providing decision makers with a comprehensive and nuanced tool to navigate complex scenarios involving diverse perspectives.

The evolution of fuzzy set theory has given rise to various extensions aimed at addressing uncertainty in more nuanced ways. Among these, the concepts of neutrosophic sets, Fermatean fuzzy sets and intuitionistic fuzzy sets have significantly contributed to the field. This literature review delves into the origins and interconnections of these theories, culminating in the formulation of Fermatean neutrosophic sets.

Fuzzy set theory, introduced by Zadeh in 1965, extends the classical notion of sets by allowing elements to have degrees of membership.[27] In a fuzzy set, each element is associated with a membership function (MF) that assigns it a value between 0 and 1, representing its degree of membership. This theory laid the groundwork for handling vagueness and imprecision in various domains. Building on fuzzy sets, Atanassov introduced Intuitionistic Fuzzy Sets (IFS) in 1986.[2] An IFS incorporates two functions: the membership function (MF) and the non-membership function (NMF), both ranging between 0 and 1. Additionally, the sum of these functions for any element does not exceed 1. This framework provides a more flexible approach by explicitly considering the hesitation margin, which reflects the uncertainty about the membership status of elements. Neutrosophic sets, proposed by Smarandache, further generalize the concept of fuzzy sets by introducing three independent functions: the membership function (MF), the non-membership function (NMF) and the indeterminacy function (IF).[22] These functions map each element to values in the interval [0, 1] and the sum of these values can range from 0 to 3. This model allows for the representation of indeterminacy, offering a more comprehensive way to handle uncertainty, contradiction and incompleteness.

Fermatean fuzzy sets were introduced by Senapati and Yager as an enhancement to traditional fuzzy sets.[21] In a Fermatean Fuzzy Set, the degrees of membership and non-membership are raised to the power of three, reflecting a more intricate relationship between these functions. This formulation aims to provide a more detailed representation of uncertainty compared to traditional fuzzy sets. The conditions governing Fermatean Fuzzy Sets ensure that the sum of the cubed values of membership and non-membership degrees remains within the unit interval. Additionally, the degree of uncertainty for an element is defined using the difference between the maximum possible value (one) and the sum of the cubed membership and non-membership degrees. This approach provides a refined mechanism for capturing the nuances of uncertainty in various scenarios. Fermatean neutrosophic sets merge the principles of Fermatean fuzzy sets and neutrosophic sets.[23] This advanced concept incorporates three functions to represent the degrees of membership, non-membership and indeterminacy, each raised to the power of three. The conditions for a Fermatean neutrosophic set ensure that the sum of the cubed values of membership and non-membership degrees is bounded and the cubed indeterminacy degree is also constrained within a specific range. This ensures a balanced representation of all three aspects of uncertainty. By combining the strengths of Fermatean fuzzy sets and neutrosophic sets, Fermatean neutrosophic Sets offer a robust framework for handling complex and multifaceted uncertainties, making them applicable to a wide range of real-world problems.

The integration of NSs and FNSs in multi-criteria decision making (MCDM) and various application domains has seen significant advancements, as evidenced by recent research. These advancements leverage specialized aggregation operators, enhancing decision making processes under uncertainty. For example, Frank's prioritization of the Bonferroni mean operator with single-valued neutrosophic sets has been pivotal in selecting third-party logistics providers.[10] This approach effectively integrates various performance metrics, addressing the inherent uncertainties in logistics provider selection and improving decision accuracy through the neutrosophic set's ability to handle indeterminacy and incomplete information. Zhou et al. proposed two novel approaches for multi-attribute group decision making using interval-valued neutrosophic Frank aggregation operators.[26] These methods accommodate incomplete weights, providing a robust framework for collective decision making when some attribute weights are unknown or partially known. This flexibility is crucial for real-world applications where complete information is often unavailable.

In the realm of Fermatean fuzzy sets, a new MCDM method has been developed by Aydon and Ozkie for selection and ranking problems.[3] This method, demonstrated through case studies, utilizes the unique properties of Fermatean fuzzy sets to better model uncertainty and hesitancy compared to traditional fuzzy sets. It offers

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a more precise tool for ranking alternatives in complex decision scenarios. Industrial 5.0 development factors have been evaluated by Lo et al. using an integration approach of Fermatean fuzzy logic.[14] This evaluation method identifies and analyzes the interrelationships between various development factors, facilitating a comprehensive understanding of how these factors interact and influence each other. This integrative approach is essential for strategic planning and development in the industrial sector. Dynamic aggregation operators have been introduced by Alghazzawi et al. for selecting optimal biometric-based attendance devices in a complex Fermatean fuzzy environment.[1] These operators enable the aggregation of dynamic and uncertain biometric data, ensuring that the selected attendance system is both reliable and adaptable to changing conditions.

A novel group decision making method has been devised for interval-valued q-rung dual hesitant fuzzy information, utilizing extended power average operators and Frank operations by Xu et al.[25] This method enhances the decision making process by capturing a wider range of hesitancy and uncertainty, making it suitable for scenarios requiring high precision and reliability in group decisions. Complex T-spherical fuzzy Frank prioritized aggregation operators have been employed in a multi-attribute decision making method by Ullah et al.[24] This innovative approach addresses the complexities of decision making in environments with spherical fuzzy data, providing a structured way to prioritize and aggregate multiple attributes effectively.

The concept of Fermatean neutrosophic has extensive applications in various fields [[5–8]]. Numerous authors have contributed to the fuzzy extension in the area of multi-criteria decision making [[11,12,15,17–19]]. In the field of artificial intelligence and military transport systems, q-spherical fuzzy rough Frank aggregation operators have been applied to AI neural networks.[4] This application demonstrates the operators' effectiveness in handling the uncertainties and complexities inherent in military logistics, leading to more efficient and reliable transport solutions. Maclaurin symmetric mean aggregation operators, based on novel Frank T-norm and T-conorm, have been utilized for picture fuzzy multiple-attribute group decision making.[20] This approach combines the strengths of picture fuzzy sets and Maclaurin symmetric mean to handle multiple attributes and group opinions, enhancing decision making processes in complex and uncertain environments. Finally, the evaluation of artificial intelligence-based solid waste segregation technologies has been conducted through multi-criteria decision making using complex q-rung picture fuzzy Frank aggregation operators.[16] This method offers a sophisticated tool for assessing and comparing various AI-based waste segregation technologies, ensuring the selection of the most efficient and sustainable solutions.

Definition 1.1. "Let Υ be the universal set containing elements known as Neutrosophic sets (NSs).[22] Each $\epsilon_\imath \in \Upsilon$ is defined as $\mathbb{N}_{\epsilon_\imath} = \{\langle \epsilon_\imath, \Psi(\epsilon_\imath), \Lambda(\epsilon_\imath), \Omega(\epsilon_\imath)\rangle | \epsilon_\imath \in \Upsilon\}$, where $\Psi(\epsilon_\imath)$, $\Omega(\epsilon_\imath)$, $\Lambda(\epsilon_\imath) : \Upsilon \to [0,1]$ represent the degrees of MF, NMF and IF of $\epsilon_\imath$. These degrees satisfy $0 \preceq \Psi(\epsilon_\imath) + \Lambda(\epsilon_\imath) + \Omega(\epsilon_\imath) \preceq 3$ for all $\epsilon_\imath \in \Upsilon$ and $i = 1, 2, 3, \ldots k$."

Definition 1.2. "Let Υ be the universal set containing elements known as Fermatean fuzzy sets (FFSs).[21] Each $\mathbb{F}_{\epsilon_\imath} \in \Upsilon$ is defined as $\mathbb{F}_{\epsilon_\imath} = \{\langle \epsilon_\imath, \Psi(\epsilon_\imath), \Omega(\epsilon_\imath)\rangle | \epsilon_\imath \in \Upsilon\}$, where $\Psi(\epsilon_\imath)$, $\Omega(\epsilon_\imath) : \Upsilon \to [0,1]$ represent the degrees of MF and NMF of $\epsilon_\imath \in \Upsilon$. These degrees satisfy $0 \preceq \Psi^3(\epsilon_\imath) + \Omega^3(\epsilon_\imath) \preceq 1$. The degree of uncertainty regarding an element $\epsilon_\imath$ is denoted as $\pi_i(\epsilon_\imath) = \sqrt[3]{1 - \Psi^3(\epsilon_\imath) - \Omega^3(\epsilon_\imath)}$ for all $\epsilon_\imath \in \Upsilon$ and $i = 1, 2, 3, \ldots k$."

Definition 1.3. "A set $\mathbb{F}_\epsilon = \{\langle \epsilon_\imath, \Psi(\epsilon_\imath), \Omega(\epsilon_\imath), \Lambda(\epsilon_\imath)\rangle | \epsilon_\imath \in \Upsilon\}$ is called Fermatean neutrosophic set (FNS)[23] in the universe of discourse Υ if $\Psi(\epsilon_\imath)$, $\Omega(\epsilon_\imath)$, $\Lambda(\epsilon_\imath) : \Upsilon \to [0,1]$ that represent the degree of MF, NMF and IF of $\epsilon_\imath \in \Upsilon$ respectively satisfy the conditions of $0 \preceq \Psi^3(\epsilon_\imath) + \Omega^3(\epsilon_\imath) \preceq 1$ and $0 \preceq \Lambda^3(\epsilon_\imath) \preceq 1$. Therefore, for FNS, $0 \preceq \Psi^3(\epsilon_\imath) + \Omega^3(\epsilon_\imath) + \Lambda^3(\epsilon_\imath) \preceq 2$ for all $\epsilon \in \Upsilon$. "

2 Spherical Fermatean Neutrosophic Sets

Definition 2.1. Let Υ be the universal set containing elements known as Spherical Fermatean Neutrosophic Sets (SFNSs). Each $\epsilon_\imath \in \Upsilon$ is defined as $\mathbb{S}_{\epsilon_\imath} = \{\langle \epsilon_\imath, \Psi(\epsilon_\imath), \Omega(\epsilon_\imath), \Lambda(\epsilon_\imath); \varrho_i\rangle : \epsilon_\imath \in \Upsilon\}$, where $\Psi_\epsilon, \Omega_\epsilon, \Lambda_\epsilon, \varrho : \Upsilon \to [0,1]$, represent the degrees of MF, NMF, IF and radius of $\epsilon_\imath$. This degrees satisfy $0 \preceq \Psi^3(\epsilon_\imath) + \Omega^3(\epsilon_\imath) \preceq 1$, $0 \preceq \Lambda^3(\epsilon_\imath) \preceq 1$ and $0 \preceq \Psi^3(\epsilon_\imath) + \Omega^3(\epsilon_\imath) + \Lambda^3(\epsilon_\iota) \preceq 2$ for all $\epsilon_\iota \in \Upsilon$ and $\imath = 1, 2, ...k$. We construct the center of the sphere by

$$\langle \Psi(\epsilon_\imath), \Omega(\epsilon_\imath), \Lambda(\epsilon_\imath)\rangle = \left\langle \frac{\sum_{\jmath=1}^{k} \Psi_{\imath,\jmath}}{k}, \frac{\sum_{\jmath=1}^{k} \Omega_{\imath,\jmath}}{k}, \frac{\sum_{\jmath=1}^{k} \Lambda_{\imath,\jmath}}{k} \right\rangle \tag{1}$$

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and the radius is

$$\varrho_i = min\left\{1 \preceq \overset{max}{j} \preceq k\ \sqrt{(\Psi(\epsilon_\imath) - \Psi\imath, \jmath)^2 + (\Omega(\epsilon_\imath) - \Omega_{\imath,\jmath})^2 + (\Lambda(\epsilon_\imath) - \Lambda_{\imath,\jmath})^2}, 1\right\}. \quad (2)$$

For example, consider $\mathbb{A} = \{\langle a, 0.3, 0.5, 0.4\rangle, \langle a, 0.4, 0.4, 0.3\rangle, \langle a, 0.5, 0.6, 0.5\rangle, \langle a, 0.6, 0.4, 0.3\rangle\}$ be the collection of FNS. Then the center of SFNS is $\langle \Psi_{\mathbb{A}}, \Omega_{\mathbb{A}}, \Lambda_{\mathbb{A}}\rangle$, where $\Psi_{\mathbb{A}} = \frac{0.3+0.4+0.5+0.6}{4} = 0.45$, $\Omega_{\mathbb{A}} = \frac{0.5+0.4+0.6+0.4}{4} = 0.48$, $\Lambda_{\mathbb{A}} = \frac{0.4+0.3+0.5+0.3}{4} = 0.38$ and the radius

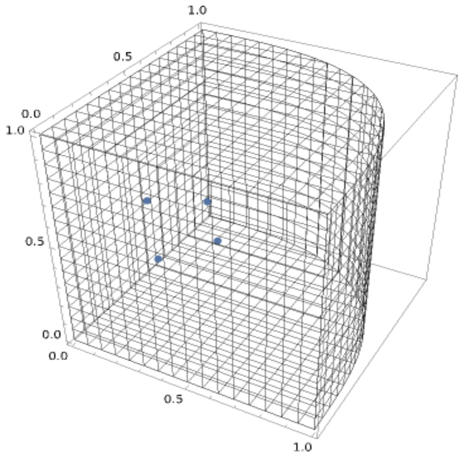
$$\varrho = min\{\max\{\sqrt{(0.45-0.3)^2 + (0.48-0.5)^2 + (0.38-0.4)^2},$$

$$\sqrt{(0.45-0.4)^2 + (0.48-0.4)^2 + (0.38-0.3)^2},$$

$$\sqrt{(0.45-0.5)^2 + (0.48-0.6)^2 + (0.38-0.5)^2},$$

$$\sqrt{(0.45-0.6)^2 + (0.48-0.4)^2 + (0.38-0.3)^2}\}, 1\}$$

$\varrho = min\{max\{0.15, 0.12, 0.18, 0.18\}, 1\} = min\{0.18, 1\} = 0.18$. The spherical Fermatean neutrosophic set is $\mathbb{S}_{\mathbb{A}} = \langle \Psi_{\mathbb{A}}, \Omega_{\mathbb{A}}, \Lambda_{\mathbb{A}}; \varrho\rangle, = \langle 0.45, 0.48, 0.38; 0.18\rangle$.



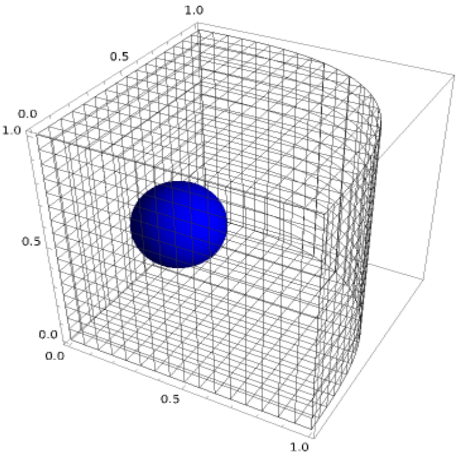


Figure 1: The geometric representation of Fermatean Neutrosophic Sets and Spherical Fermatean Neutrosophic Sets.

(The FNS $\alpha_1 = \langle 0.3, 0.5, 0.4\rangle$, $\alpha_2 = \langle 0.4, 0.4, 0.3\rangle$, $\alpha_3 = \langle 0.5, 0.6, 0.5\rangle$, $\alpha_4 = \langle 0.6, 0.4, 0.3\rangle$ and $\theta_\alpha = \langle 0.45, 0.48, 0.38; 0.18\rangle$)

Definition 2.2. Let $\theta_\alpha = \langle \Psi_\alpha, \Omega_\alpha, \Lambda_\alpha; \varrho_1\rangle$, $\theta_\beta = \langle \Psi_\beta, \Omega_\beta, \Lambda_\beta; \varrho_2\rangle$ are two s over the universal X. Then the following operations are defined as follows:

1. $\theta_\alpha \sqcup \theta_\beta = \langle max(\Psi_\alpha, \Psi_\beta), max(\Omega_\alpha, \Omega_\beta), min(\Lambda_\alpha, \Lambda_\beta), max(\varrho_1; \varrho_2)\rangle$.
2. $\theta_\alpha \sqcap \theta_\beta = \langle min(\Psi_\alpha, \Psi_\beta), min(\Omega_\alpha, \Omega_\beta), max(\Lambda_\alpha, \Lambda_\beta), min(\varrho_1; \varrho_2)\rangle$.
3. $\theta_\alpha = \theta_\beta \iff \varrho_1 = \varrho_2$ and $\Psi_\alpha = \Psi_\beta, \Omega_\alpha = \Omega_\beta, \Lambda_\alpha = \Lambda_\beta$.
4. $\theta_\alpha \sqsubset \theta_\beta \iff \varrho_1 \prec \varrho_2$ and $\Psi_\alpha \prec \Psi_\beta, \Omega_\alpha \prec \Omega_\beta, \Lambda_\alpha \succ \Lambda_\beta$.
5. $\theta_\alpha \sqsupset \theta_\beta \iff \varrho_1 \succ \varrho_2$ and $\Psi_\alpha \succ \Psi_\beta, \Omega_\alpha \succ \Omega_\beta, \Lambda_\alpha \prec \Lambda_\beta$.
6. $\theta_\alpha \boxplus \theta_\beta = \langle \Psi_\alpha + \Psi_\beta - \Psi_\alpha\Psi_\beta, \Omega_\alpha\Omega_\beta, \Lambda_\alpha\Lambda_\beta; \varrho_1 + \varrho_2 - \varrho_1\varrho_2\rangle$.
7. $\theta_\alpha \boxtimes \theta_\beta = \langle \Psi_\alpha\Psi_\beta, \Omega_\alpha + \Omega_\beta - \Omega_\alpha\Omega_\beta, \Lambda_\alpha + \Lambda_\beta - \Lambda_\alpha\Lambda_\beta; \varrho_1\varrho_2\rangle$.
8. $\zeta\theta_\alpha = \left\langle 1 - (1-\Psi_\alpha)^\zeta, (\Omega_\alpha)^\zeta, (\Lambda_\alpha)^\zeta, 1 - (1-\varrho_1)^\zeta\right\rangle$.
9. $\theta_\alpha^\zeta = \left\langle \Psi_\alpha^\zeta, 1 - (1-\Omega_\alpha)^\zeta, 1 - (1-\Lambda_\alpha)^\zeta; \varrho_1^\zeta\right\rangle$.

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10. $\neg\theta = \langle \Lambda_\alpha, \Omega_\alpha, \Psi_\alpha; \varrho_1 \rangle$.
11. $\theta^c = \langle \Lambda_\alpha, \Omega_\alpha, \Psi_\alpha; 1-\varrho_1 \rangle$.

Example 2.3. Let $\alpha_1 = \langle 0.3, 0.5, 0.4 \rangle$, $\alpha_2 = \langle 0.4, 0.4, 0.3 \rangle$, $\alpha_3 = \langle 0.5, 0.6, 0.5 \rangle$, $\alpha_4 = \langle 0.6, 0.4, 0.3 \rangle$, $\beta_1 = \langle 0.1, 0.4, 0.5 \rangle$, $\beta_2 = \langle 0.2, 0.3, 0.3 \rangle$, $\beta_3 = \langle 0.4, 0.4, 0.4 \rangle$, $\beta_4 = \langle 0.3, 0.5, 0.5 \rangle$, are FNSs, SFNSs are $\theta_\alpha = \langle 0.45, 0.48, 0.38; 0.18 \rangle$, $\zeta = 0.5$ and $\theta_\beta = \langle 0.25, 0.4, 0.43; 0.17 \rangle$. Then

1. $\theta_\alpha \sqcup \theta_\beta = \langle 0.45, 0.48, 0.38; 0.18 \rangle$.
2. $\theta_\alpha \sqcap \theta_\beta = \langle 0.25, 0.4, 0.43; 0.17 \rangle$.
3. $\theta_\alpha \boxplus \theta_\beta = \langle 0.59, 0.19, 0.16; 0.32 \rangle$.
4. $\theta_\alpha \boxtimes \theta_\beta = \langle 0.11, 0.69, 0.65; 0.03 \rangle$.
5. $\zeta\theta_\alpha = \langle 0.26, 0.69, 0.62; 0.09 \rangle$.
6. $\theta_\zeta^\alpha = \langle 0.67, 0.28, 0.55; 0.42 \rangle$.
7. $\neg\theta_\alpha = \langle 0.38, 0.48, 0.45; 0.18 \rangle$.
8. $\theta_\alpha^c = \langle 0.38, 0.48, 0.45; 0.82 \rangle$.

Proposition 2.4. *For any three SFNSs $\theta_1, \theta_2, \theta_3$, the following results are valid.*

1. $\theta_1 \sqcap \theta_2 = \theta_2 \sqcap \theta_1$ *and* $\theta_1 \sqcup \theta_2 = \theta_2 \sqcup \theta_1$.
2. $(\theta_1 \sqcap \theta_2) \sqcap \theta_3 = \theta_1 \sqcap (\theta_2 \sqcap \theta_3)$ *and* $(\theta_1 \sqcup \theta_2) \sqcup \theta_3 = \theta_1 \sqcup (\theta_2 \sqcup \theta_3)$.
3. $(\theta_1 \sqcup \theta_2) \sqcap \theta_3 = (\theta_1 \sqcap \theta_3) \sqcup (\theta_2 \sqcap \theta_3)$ *and* $(\theta_1 \sqcap \theta_2) \sqcup \theta_3 = (\theta_1 \sqcup \theta_3) \sqcap (\theta_2 \sqcup \theta_3)$.
4. $\neg(\neg\theta_1 \sqcap \neg\theta_2) = \theta_1 \sqcup \theta_2$ *and* $\neg(\neg\theta_1 \sqcup \neg\theta_2) = \theta_1 \sqcap \theta_2$.
5. $\theta_1 \sqcap \theta_1 = \theta_1$ *and* $\theta_1 \sqcup \theta_1 = \theta_1$.
6. $\theta_1 \boxplus \theta_2 = \theta_2 \boxplus \theta_1$ *and* $\theta_1 \boxtimes \theta_2 = \theta_2 \boxtimes \theta_1$.
7. $(\theta_1 \boxplus \theta_2) \boxplus \theta_3 = \theta_1 \boxplus (\theta_2 \boxplus \theta_3)$ *and* $(\theta_1 \boxtimes \theta_2) \boxtimes \theta_3 = \theta_1 \boxtimes (\theta_2 \boxtimes \theta_3)$.
8. $(\theta_1 \sqcap \theta_2) \boxplus \theta_3 = (\theta_1 \boxplus \theta_3) \sqcap (\theta_2 \boxplus \theta_3)$ *and* $(\theta_1 \sqcap \theta_2) \boxtimes \theta_3 = (\theta_1 \boxtimes \theta_3) \sqcap (\theta_2 \boxtimes \theta_3)$.
9. $(\theta_1 \sqcup \theta_2) \boxplus \theta_3 = (\theta_1 \boxplus \theta_3) \sqcup (\theta_2 \boxplus \theta_3)$ *and* $(\theta_1 \sqcup \theta_2) \boxtimes \theta_3 = (\theta_1 \boxtimes \theta_3) \sqcup (\theta_2 \boxtimes \theta_3)$.
10. $\neg(\neg\theta_1 \boxplus \neg\theta_2) = \theta_1 \boxtimes \theta_2$ *and* $\neg(\neg\theta_1 \boxtimes \neg\theta_2) = \theta_1 \boxplus \theta_2$.

3 Spherical Fermatean Neutrosophic Frank Aggregation Operator

Definition 3.1. Let $\alpha_\imath = \langle \Psi_{\alpha_\imath}, \Omega_{\alpha_\imath}, \Lambda_{\alpha_\imath}; \varrho_{\alpha_\imath} \rangle ; \imath \in \mathbb{Z}^+$ be a number of $SFNSs$, then the aggregated value of them using $SFNSs$ weighted averaging operator also a

$$SFNWA(\alpha_1, \alpha_2, \alpha_3, ...\alpha_n) = \begin{cases} \sqrt[3]{1 - \prod_{\imath=1}^n (1 - \Psi_{\alpha_\imath}^3)^{\omega_\imath}}, \prod_{\imath=1}^n \Omega_{\alpha_i}^{3^{\omega_\imath}}, \\ \prod_{\imath=1}^n \Lambda_{\alpha_i}^{3^{\omega_\imath}}; \sqrt[3]{1 - \prod_{\imath=1}^n (1 - \varrho_{\alpha_\imath}^3)^{\omega_\imath}} \end{cases}$$

where $\omega = (\omega_1, \omega_2, \omega_3.....\omega_n)^t$ be the weight vector of $\alpha_\imath; k \in Z_n^+, \omega_\imath \in [0,1]$ and $\sum_{\imath=1}^n \omega_\imath = 1$.

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Definition 3.2. Let $\alpha =< \Psi_\alpha, \Omega_\alpha, \Lambda_\alpha; \varrho_\alpha >$ and $\beta =< \Psi_\beta, \Omega_\beta, \Lambda_\beta; \varrho_\beta >$ be any two SFNSs, $\bigcirc > 1$. Then Frank t-norm and t-conorm operations of SFNSs for any real $\theta > 0$ defined as

$$1.\ \alpha \oplus \beta = \begin{cases} \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)(\bigcirc^{(1-\Psi_\beta^3)}-1)}{\bigcirc-1}\right]}, \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Omega_\alpha^3}-1)(\bigcirc^{\Omega_\beta^3}-1)}{\bigcirc-1}\right], \\ \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Lambda_\alpha^3}-1)(\bigcirc^{\Lambda_\beta^3}-1)}{\bigcirc-1}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)(\bigcirc^{(1-\varrho_\beta^3)}-1)}{\bigcirc-1}\right]} \end{cases}$$

$$2.\ \alpha \otimes \beta = \begin{cases} \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Psi_\alpha^3}-1)(\bigcirc^{\Psi_\beta^3}-1)}{\bigcirc-1}\right], \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Omega_\alpha^3)}-1)(\bigcirc^{(1-\Omega_\beta^3)}-1)}{\bigcirc-1}\right]}, \\ \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Lambda_\alpha^3)}-1)(\bigcirc^{(1-\Lambda_\beta^3)}-1)}{\bigcirc-1}\right]}; \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\varrho_\alpha^3}-1)(\bigcirc^{\varrho_\beta^3}-1)}{\bigcirc-1}\right]. \end{cases}$$

$$3.\ \theta\alpha = \begin{cases} \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right]}, \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Omega_\alpha^3}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right], \\ \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Lambda_\alpha^3}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right]}. \end{cases}$$

$$4.\ \alpha^\theta = \begin{cases} \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Psi_\alpha^3}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right], \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Omega_\alpha^3)}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right]}, \\ \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Lambda_\alpha^3)}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right]}; \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\varrho_\alpha^3}-1)^\theta}{(\bigcirc-1)^{\theta-1}}\right]. \end{cases}$$

Theorem 3.3. *Let $\alpha = (\Psi_\alpha, \Omega_\alpha, \Lambda_\alpha; \varrho_\alpha)$ and $\beta = (\Psi_\beta, \Omega_\beta, \Lambda_\beta; \varrho_\beta)$ be any two SFNNs. $\theta, \gamma \in \mathcal{Z}^+$, Then*

1. $\alpha \oplus \beta = \beta \oplus \alpha$.
2. $\alpha \otimes \beta = \beta \otimes \alpha$.
3. $\theta(\alpha \oplus \beta) = (\theta\alpha) \oplus (\theta\beta)$.
4. $\theta\alpha \oplus \gamma\alpha = (\theta + \gamma)\alpha$.
5. $(\alpha \otimes \beta)^\theta = \alpha^\theta \otimes \beta^\theta$.
6. $\alpha^\theta \otimes \alpha^\gamma = \alpha^{(\theta+\gamma)}$.

Proof. *For two FNFNs $\alpha = (\Psi_\alpha, \Omega_\alpha, \Lambda_\alpha; \varrho_\alpha)$, $\beta = (\Psi_\beta, \Omega_\beta, \Lambda_\beta; \varrho_\beta)$ and $\theta, \gamma > 0$,*

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$$1.\ \alpha \oplus \beta = \left\{ \begin{array}{l} \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Psi_{\alpha}^{3})}-1)(\bigcirc^{(1-\Psi_{\beta}^{3})}-1)}{\bigcirc-1}\right]}, \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Omega_{\alpha}^{3}}-1)(\bigcirc^{\Omega_{\beta}^{3}}-1)}{\bigcirc-1}\right], \\ \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Lambda_{\alpha}^{3}}-1)(\bigcirc^{\Lambda_{\beta}^{3}}-1)}{\bigcirc-1}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\varrho_{\alpha}^{3})}-1)(\bigcirc^{(1-\varrho_{\beta}^{3})}-1)}{\bigcirc-1}\right]} \end{array} \right.$$

$$= \left\{ \begin{array}{l} \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Psi_{\beta}^{3})}-1)(\bigcirc^{(1-\Psi_{\alpha}^{3})}-1)}{\bigcirc-1}\right.}, \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Omega_{\beta}^{3}}-1)(\bigcirc^{\Omega_{\alpha}^{3}}-1)}{\bigcirc-1}\right], \\ \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Lambda_{\beta}^{3}}-1)(\bigcirc^{\Lambda_{\alpha}^{3}}-1)}{\bigcirc-1}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\varrho_{\beta}^{3})}-1)(\bigcirc^{(1-\varrho_{\alpha}^{3})}-1)}{\bigcirc-1}\right.} \end{array} \right.$$

$$=\beta \oplus \alpha.$$

$$2.\ \alpha \otimes \beta = \left\{ \begin{array}{l} \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Psi_{\alpha}^{3}}-1)(\bigcirc^{\Psi_{\beta}^{3}}-1)}{\bigcirc-1}\right], \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Omega_{\alpha}^{3})}-1)(\bigcirc^{(1-\Omega_{\beta}^{3})}-1)}{\bigcirc-1}\right]}, \\ \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Lambda_{\alpha}^{3})}-1)(\bigcirc^{(1-\Lambda_{\beta}^{3})}-1)}{\bigcirc-1}\right]}; \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\varrho_{\alpha}^{3}}-1)(\bigcirc^{\varrho_{\beta}^{3}}-1)}{\bigcirc-1}\right] \end{array} \right.$$

$$= \left\{ \begin{array}{l} \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Psi_{\beta}^{3}}-1)(\bigcirc^{\Psi_{\alpha}^{3}}-1)}{\bigcirc-1}\right], \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Omega_{\beta}^{3})}-1)(\bigcirc^{(1-\Omega_{\alpha}^{3})}-1)}{\bigcirc-1}\right]}, \\ \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Lambda_{\beta}^{3})}-1)(\bigcirc^{(1-\Lambda_{\alpha}^{3})}-1)}{\bigcirc-1}\right]}; \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\varrho_{\beta}^{3}}-1)(\bigcirc^{\varrho_{\alpha}^{3}}-1)}{\bigcirc-1}\right] \end{array} \right.$$

$$=\beta \otimes \alpha.$$

$$3.\ \theta(\alpha \oplus \beta) = \left\{ \begin{array}{l} \theta\left\{ \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Psi_{\alpha}^{3})}-1)(\bigcirc^{(1-\Psi_{\beta}^{3})}-1)}{\bigcirc-1}\right]}, \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Omega_{\alpha}^{3}}-1)(\bigcirc^{\Omega_{\beta}^{3}}-1)}{\bigcirc-1}\right], \right. \\ \left. \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Lambda_{\alpha}^{3}}-1)(\bigcirc^{\Lambda_{\beta}^{3}}-1)}{\bigcirc-1}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\varrho_{\alpha}^{3})}-1)(\bigcirc^{(1-\varrho_{\beta}^{3})}-1)}{\bigcirc-1}\right]} \right\} \end{array} \right.$$

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$$=\left\{\begin{array}{l}\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)^\theta(\bigcirc^{(1-\Psi_\beta^3)}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right]},\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Omega_\alpha^3}-1)^\theta(\bigcirc^{\Omega_\beta^3}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right],\\ \log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Lambda_\alpha^3}-1)^\theta(\bigcirc^{\Lambda_\beta^3}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)^\theta(\bigcirc^{(1-\varrho_\beta^3)}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right]}\end{array}\right.$$

$$=(\theta\alpha)\oplus(\theta\beta)$$

$$=\left\{\begin{array}{l}\left\{\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)^\theta}{(\bigcirc-1)^\theta}\right]},\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Omega_\alpha^3}-1)^\theta}{(\bigcirc-1)^\theta}\right],\right.\\ \left.\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Lambda_\alpha^3}-1)^\theta}{(\bigcirc-1)^\theta}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)^\theta}{(\bigcirc-1)^\theta}\right]}\right\}\oplus\\ \left\{\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Psi_\beta^3)}-1)^\theta}{(\bigcirc-1)^\theta}\right]},\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Omega_\beta^3}-1)^\theta}{(\bigcirc-1)^\theta}\right],\right.\\ \left.\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Lambda_\beta^3}-1)^\theta}{(\bigcirc-1)^\theta}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\varrho_\beta^3)}-1)^\theta}{(\bigcirc-1)^\theta}\right]}\right\}\end{array}\right.$$

$$=\left\{\begin{array}{l}\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)^\theta(\bigcirc^{(1-\Psi_\beta^3)}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right]},\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Omega_\alpha^3}-1)^\theta(\bigcirc^{\Omega_\beta^3}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right],\\ \log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Lambda_\alpha^3}-1)^\theta(\bigcirc^{\Lambda_\beta^3}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)^\theta(\bigcirc^{(1-\varrho_\beta^3)}-1)^\theta}{(\bigcirc-1)^{2\theta-1}}\right]}\end{array}\right.$$

$$\theta(\alpha\oplus\beta)=(\theta\alpha)\oplus(\theta\beta)$$

$$4.\ \theta\alpha\oplus\gamma\alpha=\left\{\begin{array}{l}\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)^\theta}{(\bigcirc-1)^\theta}\right]},\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Omega_\alpha^3}-1)^\theta}{(\bigcirc-1)^\theta}\right],\\ \left.\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Lambda_\alpha^3}-1)^\theta}{(\bigcirc-1)^\theta}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)^\theta}{(\bigcirc-1)^\theta}\right]}\right\}\oplus\\ \left\{\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)^\gamma}{(\bigcirc-1)^\gamma}\right]},\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Omega_\alpha^3}-1)^\gamma}{(\bigcirc-1)^\gamma}\right],\right.\\ \log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Lambda_\alpha^3}-1)^\gamma}{(\bigcirc-1)^\gamma}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)^\gamma}{(\bigcirc-1)^\gamma}\right]}\end{array}\right.$$

$$=\left\{\begin{array}{l}\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Psi_\alpha^3)}-1)^{\theta+\gamma}}{(\bigcirc-1)^{\theta+\gamma}}\right]},\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Omega_\alpha^3}-1)^{\theta+\gamma}}{(\bigcirc-1)^{\theta+\gamma}}\right],\\ \log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Lambda_\alpha^3}-1)^{\theta+\gamma}}{(\bigcirc-1)^{\theta+\gamma}}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\varrho_\alpha^3)}-1)^{\theta+\gamma}}{(\bigcirc-1)^{\theta+\gamma}}\right]}\end{array}\right.$$

$$=(\theta+\gamma)\alpha$$

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$$5.\ (\alpha\otimes\beta)^{\kappa}=\left\{\begin{array}{l}\left\{\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Psi_{\alpha}^{3}}-1)(\bigcirc^{\Psi_{\beta}^{3}}-1)}{\bigcirc-1}\right],\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Omega_{\alpha}^{3})}-1)(\bigcirc^{(1-\Omega_{\beta}^{3})}-1)}{\bigcirc-1}\right]},\right.\\ \left.\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Lambda_{\alpha}^{3})}-1)(\bigcirc^{(1-\Lambda_{\beta}^{3})}-1)}{\bigcirc-1}\right]};\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\varrho_{\alpha}^{3}}-1)(\bigcirc^{\varrho_{\beta}^{3}}-1)}{\bigcirc-1}\right],\right\}^{\kappa}\end{array}\right.$$

$$=\left\{\begin{array}{l}\log_{\bigcirc}\left[1+\frac{((\bigcirc^{\Psi_{\alpha}^{3}}-1)(\bigcirc^{\Psi_{\beta}^{3}}-1))^{\kappa}}{(\bigcirc-1)^{2\kappa-1}}\right],\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{((\bigcirc^{(1-\Omega_{\alpha}^{3})}-1)(\bigcirc^{(1-\Omega_{\beta}^{3})}-1))^{\kappa}}{(\bigcirc-1)^{2\kappa-1}}\right]},\\ \sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{((\bigcirc^{(1-\Lambda_{\alpha}^{3})}-1)(\bigcirc^{(1-\Lambda_{\beta}^{3})}-1))^{\kappa}}{(\bigcirc-1)^{2\kappa-1}}\right]};\log_{\bigcirc}\left[1+\frac{((\bigcirc^{\varrho_{\alpha}^{3}}-1)(\bigcirc^{\varrho_{\beta}^{3}}-1))^{\kappa}}{(\bigcirc-1)^{2\kappa-1}}\right]\end{array}\right.$$

$$(\alpha\otimes\beta)^{\kappa}=\left\{\begin{array}{l}\left\{\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Psi_{\alpha}^{3}}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right],\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Omega_{\alpha}^{3})}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right]},\right.\\ \left.\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Lambda_{\alpha}^{3})}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right]};\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\varrho_{\alpha}^{3}}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right]\right\}\otimes\\ \left\{\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Psi_{\beta}^{3}}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right],\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Omega_{\beta}^{3})}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right]},\right.\\ \left.\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Lambda_{\beta}^{3})}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right]};\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\varrho_{\beta}^{3}}-1)^{\kappa}}{(\bigcirc-1)^{\kappa}}\right]\right\}\end{array}\right.$$

$$=\alpha^{\kappa}\otimes\beta^{\kappa}.$$

$$6.\ \alpha^{\kappa}\otimes\alpha^{\gamma}=\left\{\begin{array}{l}\left\{\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Psi_{\alpha}^{3}}-1)^{\kappa}}{(\bigcirc-1)^{\kappa-1}}\right],\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Omega_{\alpha}^{3})}-1)^{\kappa}}{(\bigcirc-1)^{\kappa-1}}\right]},\right.\\ \left.\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Lambda_{\alpha}^{3})}-1)^{\kappa}}{(\bigcirc-1)^{\kappa-1}}\right]};\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\varrho_{\alpha}^{3}}-1)^{\kappa}}{(\bigcirc-1)^{\kappa-1}}\right]\right\}\otimes\\ \left\{\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Psi_{\beta}^{3}}-1)^{\gamma}}{(\bigcirc-1)^{\gamma-1}}\right],\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Omega_{\beta}^{3})}-1)^{\gamma}}{(\bigcirc-1)^{\gamma-1}}\right]},\right.\\ \left.\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Lambda_{\beta}^{3})}-1)^{\gamma}}{(\bigcirc-1)^{\gamma-1}}\right]};\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\varrho_{\beta}^{3}}-1)^{\gamma}}{(\bigcirc-1)^{\gamma-1}}\right]\right\}\end{array}\right.$$

$$=\left\{\begin{array}{l}\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\Psi_{\alpha}^{3}}-1)^{\kappa+\gamma}}{(\bigcirc-1)^{\kappa+\gamma-1}}\right],\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{(\bigcirc^{(1-\Omega_{\alpha}^{3})}-1)^{\kappa+\gamma}}{(\bigcirc-1)^{\kappa+\gamma-1}}\right]}\\ \sqrt[3]{1-\log_{\bigcirc}\left[(1+\frac{(\bigcirc^{(1-\Lambda_{\alpha}^{3})}-1)^{\kappa+\gamma}}{(\bigcirc-1)^{\kappa+\gamma-1}}\right]};\log_{\bigcirc}\left[1+\frac{(\bigcirc^{\varrho_{\alpha}^{3}}-1)^{\kappa+\gamma}}{(\bigcirc-1)^{\kappa+\gamma-1}}\right]\end{array}\right.$$

$$=\alpha^{\kappa+\gamma}.$$

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4 Fermatean Neutroshopic Frank Arithmetic Aggregation Operator

Definition 4.1. Let $\alpha_\imath = (\Psi_{\alpha_\imath}, \Omega_{\alpha_\imath}, \Lambda_{\alpha_\imath}; \varrho_{\alpha_\imath})(\imath = 1, 2, 3, ...n)$ be a number of $SFNNs$. Then Spherical Fermatean Neutroshopic Frank Weighted Arithmetic Aggregation ($\mathbb{SFNFA}$) operator is

$$\mathbb{SFNFA}(\alpha_1, \alpha_2, ...\alpha_n) = \bigoplus_{\imath=1}^{n} \omega_\imath \alpha_\imath$$

where $w = (\omega_1, \omega_2, \omega_3.....\omega_n)^t$ be the weight vector of $\alpha_\imath; k \in Z_n^+, \omega_\imath \in [0, 1]$ and $\sum_{\imath=1}^{n} \omega_\imath = 1.$

Theorem 4.2. *Let $\alpha_\imath = (\Psi_{\alpha_\imath}, \Omega_{\alpha_\imath}, \Lambda_{\alpha_\imath}; \varrho_{\alpha_\imath})(\imath = 1, 2, 3....n)$ be a $SFNNs$, then aggregated value of them using $\mathbb{SFNFA}$ operator is a $SFNNs$ and*

$$\begin{aligned}\mathbb{SFNFA}(\alpha_1, \alpha_2, ...\alpha_n) =& \bigoplus_{\imath=1}^{n} \omega_\imath \alpha_\imath \\ =& \left\{\begin{array}{l} \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{n}(\bigcirc^{(1-\Psi^3_{\alpha_\imath})} - 1)^{\omega_\imath}\right]}, \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{n}(\bigcirc^{\Omega^3_{\alpha_\imath}} - 1)^{\omega_\imath}\right], \\ \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{n}(\bigcirc^{\Lambda^3_{\alpha_\imath}} - 1)^{\omega_\imath}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{n}(\bigcirc^{(1-\varrho^3_{\alpha_\imath})} - 1)^{\omega_\imath}\right]} \end{array}\right.\end{aligned}$$

Proof: By mathematical induction. Let $n = 2$.

$$\begin{aligned}\mathbb{SFNFA}(\alpha_1, \alpha_2) =& \bigoplus_{\imath=1}^{2} \omega_\imath \alpha_i \\ =& \omega_1\alpha_1 \oplus \omega_2\alpha_2\end{aligned}$$

$$= \left\{\begin{array}{l} \left\{\sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Psi^3_{\alpha_1})} - 1)^{\omega_1}}{(\bigcirc - 1)^{\omega_1 - 1}}\right]}, \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Omega^3_{\alpha_1}} - 1)^{\omega_1}}{(\bigcirc - 1)^{\omega_1 - 1}}), \right] \right. \\ \left. \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Lambda^3_{\alpha_1}} - 1)^{\omega_1}}{(\bigcirc - 1)^{\omega_1 - 1}}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\varrho^3_{\alpha_1})} - 1)^{\omega_1}}{(\bigcirc - 1)^{\omega_1 - 1}}\right]}\right\} \oplus \\ \left\{\sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\Psi^3_{\alpha_2})} - 1)^{\omega_2}}{(\bigcirc - 1)^{\omega_2 - 1}}\right]}, \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Omega^3_{\alpha_2}} - 1)^{\omega_2}}{(\bigcirc - 1)^{\omega_2 - 1}}\right], \right. \\ \left. \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{\Lambda^3_{\alpha_2}} - 1)^{\omega_2}}{(\bigcirc - 1)^{\omega_2 - 1}}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \frac{(\bigcirc^{(1-\varrho^3_{\alpha_2})} - 1)^{\omega_2}}{(\bigcirc - 1)^{\omega_2 - 1}}\right]}\right\} \end{array}\right.$$

$$= \left\{\begin{array}{l} \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{2}(\bigcirc^{(1-\Psi^3_{\alpha_\imath})} - 1)^{\omega_\imath}\right]}, \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{2}(\bigcirc^{\Omega^3_{\alpha_\imath}} - 1)^{\omega_\imath}\right], \\ \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{2}(\bigcirc^{\Lambda^3_{\alpha_\imath}} - 1)^{\omega_\imath}\right]; \sqrt[3]{1 - \log_{\bigcirc}\left[1 + \prod_{\imath=1}^{2}(\bigcirc^{(1-\varrho^3_{\alpha_\imath})} - 1)^{\omega_\imath}\right]} \end{array}\right.$$

It is true for n=2. Assume the result is true for n=m.

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$$\mathbb{SFNFA}(\alpha_1,\alpha_2,...\alpha_m)=\bigoplus_{\imath=1}^{m}\omega_\imath\alpha_\imath$$
$$=\begin{cases}\left\{\sqrt[3]{1-\log_{\bigcirc}\left[1+\prod_{\imath=1}^{m}(\bigcirc^{(1-\Psi^3_{\alpha_\imath})}-1)^{\omega_\imath}\right]},\log_{\bigcirc}\left[1+\prod_{\imath=1}^{m}(\bigcirc^{\Omega^3_{\alpha_\imath}}-1)^{\omega_\imath}\right],\right.\\ \left.\log_{\bigcirc}\left[1+\prod_{\imath=1}^{m}(\bigcirc^{\Lambda^3_{\alpha_\imath}}-1)^{\omega_\imath}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\prod_{\imath=1}^{m}(\bigcirc^{(1-\varrho^3_{\alpha_\imath})}-1)^{\omega_\imath}\right]}\right\}\end{cases}$$

we prove for $n=m+1$. $\mathbb{SFNFA}(\alpha_1,\ldots,\alpha_{m+1})$

$$=\bigoplus_{\imath=1}^{m+1}\omega_\imath\alpha_\imath=\bigoplus_{\imath=1}^{m}\omega_\imath\alpha_\imath\bigoplus\omega_{m+1}\alpha_{m+1}$$
$$=\begin{cases}\left\{\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{(1-\Psi^3_{\alpha_\imath})}-1)^{\omega_\imath}}{(\bigcirc-1)^{\sum_{\imath=1}^{m}\omega_\imath-1}}\right]},\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{\Omega^3_{\alpha_\imath}}-1)^{\omega_\imath}}{(\bigcirc-1)^{\sum_{\imath=1}^{m}\omega_\imath-1}}),\right]\right.\\ \left.\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{\Lambda^3_{\alpha_\imath}}-1)^{\omega_\imath}}{(\bigcirc-1)^{\sum_{\imath=1}^{s}\omega_\imath-1}}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{(1-\varrho^3_{\alpha_\imath})}-1)^{\omega_\imath}}{(\bigcirc-1)^{\sum_{\imath=1}^{m}\omega_\imath-1}}\right]}\right\}\oplus\\ \left\{\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{(1-\Psi^3_{\alpha_{m+1}})}-1)^{\omega_{m+1}}}{(\bigcirc-1)^{\omega_{m+1}-1}}\right]},\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{\Omega^3_{\alpha_{m+1}}}-1)^{\omega_{m+1}}}{(\bigcirc-1)^{\omega_{m+1}-1}}\right],\right.\\ \left.\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{\Lambda^3_{\alpha_{m+1}}}-1)^{\omega_{m+1}}}{(\bigcirc-1)^{\omega_{m+1}-1}}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\frac{\prod_{\imath=1}^{m}(\bigcirc^{(1-\varrho^3_{\alpha_{m+1}})}-1)^{\omega_{m+1}}}{(\bigcirc-1)^{\omega_{m+1}-1}}\right]}\right\}\end{cases}$$
$$=\begin{cases}\sqrt[3]{1-\log_{\bigcirc}\left[1+\prod_{\imath=1}^{m+1}(\bigcirc^{(1-\Psi^3_{\alpha_\imath})}-1)^{\omega_\imath}\right]},\log_{\bigcirc}\left[1+\prod_{\imath=1}^{m+1}(\bigcirc^{\Omega^3_{\alpha_\imath}}-1)^{\omega_\imath}\right],\\ \log_{\bigcirc}\left[1+\prod_{\imath=1}^{m+1}(\bigcirc^{\Lambda^3_{\alpha_\imath}}-1)^{\omega_\imath}\right];\sqrt[3]{1-\log_{\bigcirc}\left[1+\prod_{\imath=1}^{m+1}(\bigcirc^{(1-\varrho^3_{\alpha_\imath})}-1)^{\omega_\imath}\right]}\end{cases}$$

Therefor by mathematical induction, the result is valid for any natural number.

Theorem 4.3. *(Idempotency property) If* $\alpha_\imath=(\Psi_{\alpha_\imath},\Omega_{\alpha_\imath},\Lambda_{\alpha_\imath};\varrho_{\alpha_\imath})(\imath=1,2,3,....n)$ *be the collection of identical* $SFNNs$*. i.e.* $\alpha_\imath$=α *for all* k, *where* α= $(\Psi_\alpha,\Omega_\alpha,\Psi_\alpha;\varrho_\alpha)$, *then* $\mathbb{SFNFA}\{\alpha_\imath;\imath=1,2,\ldots n\}$ = α.

Proof: *Given* $\alpha_\imath$=α *for all* k, *then* ,

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$$\mathbb{SFNFA}(\alpha_1,\alpha_2,\ldots\alpha_n)=\begin{cases}\sqrt[3]{1-log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{(1-\Psi^3_{\alpha_\imath})}-1)^{\omega_\imath}\right]}, log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{\Omega^3_{\alpha_\imath}}-1)^{\omega_\imath}\right],\\ log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{\Lambda^3_{\alpha_\imath}}-1)^{\omega_\imath}\right]; \sqrt[3]{1-log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{(1-\varrho^3_{\alpha_\imath})}-1)^{\omega_\imath}\right]}\end{cases}$$

$$=\begin{cases}\sqrt[3]{1-log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{(1-\Psi^3_{\alpha})}-1)^{\omega_\imath}\right]}, log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{\Omega^3_{\alpha}}-1)^{\omega_\imath}\right],\\ log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{\Lambda^3_{\alpha}}-1)^{\omega_\imath}\right]; \sqrt[3]{1-log_{\bigcirc}\left[1+\prod_{\imath=1}^{n}(\bigcirc^{(1-\varrho^3_{\alpha})}-1)^{\omega_\imath}\right]}\end{cases}$$

$$=\begin{cases}\sqrt[3]{1-log_{\bigcirc}\left[1+(\bigcirc^{(1-\Psi^3_{\alpha})}-1)^{\sum_{\imath=1}^{n}\omega_\imath}\right]}, log_{\bigcirc}\left[1+(\bigcirc^{\Omega^3_{\alpha}}-1)^{\sum_{\imath=1}^{n}\omega_\imath}\right],\\ log_{\bigcirc}\left[1+(\bigcirc^{\Lambda^3_{\alpha}}-1)^{\sum_{\imath=1}^{n}\omega_i}\right]; \sqrt[3]{1-log_{\bigcirc}\left[1+(\bigcirc^{(1-\varrho^3_{\alpha})}-1)^{\sum_{\imath=1}^{n}\omega_\imath}\right]}\end{cases}$$

$$=\begin{cases}\sqrt[3]{1-log_{\bigcirc}\left[1+(\bigcirc^{(1-\Psi^3_{\alpha})}-1)\right]}, log_{\bigcirc}\left[1+(\bigcirc^{\Omega^3_{\alpha}}-1)\right],\\ log_{\bigcirc}\left[1+(\bigcirc^{\Lambda^3_{\alpha}}-1)\right]; \sqrt[3]{1-log_{\bigcirc}\left[1+(\bigcirc^{(1-\varrho^3_{\alpha})}-1)\right]}\end{cases}$$

$\alpha_\imath=(\Psi_{\alpha_\imath},\Omega_{\alpha_\imath},\Lambda_{\alpha_\imath};\varrho_{\alpha_\imath})=\alpha$.

Theorem 4.4. *(Boundedness property) Let* $\alpha_\imath=(\Psi_{\alpha_\imath},\Omega_{\alpha_\imath},\Lambda_{\alpha_\imath};\varrho_{\alpha_\imath})(\imath=1,2,3,....n)$ *be a collection of identical* $SFNSs$. *Let* $\alpha^-=$ *minimum of* $\{\alpha_\imath;\imath=1,2,3.....n\}$ *and* $\alpha^+=$ *maximum of* $\{\alpha_\imath;\imath=1,2,....n\}$. *Then* $\alpha^-\leq\mathbb{SFNFA}(\alpha_1,\alpha_2,\ldots\alpha_n)\leq\alpha^+$

5 Multi-Criteria Decision Making Using Spherical Fermatean Neutrosophic Sets

In this section, we propose a Multi-Criteria Decision Making (MCDM) approach using the Spherical Fermatean Neutrosophic Frank aggregation operator. When multiple decision makers are involved in the decision making process, simply averaging decision values may not accurately represent their collective perspective. The Spherical Fermatean Neutrosophic addresses the limitations of traditional averaging methods. We apply this approach to evaluate the usefulness of emerging technology commercialization.

Let $\mathbb{A}=\{\mathbb{A}_1,\mathbb{A}_2\ldots\mathbb{A}_\lambda\}$ be a set of alternatives and $\mathbb{C}=\{\mathbb{C}_1,\mathbb{C}_2\ldots\mathbb{C}_\lambda\}$ be a set of criteria. Suppose $(\delta_{\alpha\varepsilon})_{m\times n}=\langle\Psi_{\delta_\alpha},\Omega_{\delta_\alpha},\Lambda_{\delta_\alpha}\rangle_{m\times n}$ is a Fermatean Neutrosophic decision matrix, where Ψ_{δ_α} is the degree of membership of alternatives $\mathbb{A}_\epsilon$, Ω_{δ_α} is the degree of neutral membership of alternatives $\mathbb{A}_\epsilon$, and Λ_{δ_α} is the degree non-membership of alternatives $\mathbb{A}_\epsilon$, each alternatives $\mathbb{A}_\epsilon$ satisfy $0\preceq\Psi^3_{\delta_\alpha}+\Omega^3_{\delta_\alpha}\preceq1$ and $0\preceq\Lambda^3_{\delta_\alpha}\preceq1$. Therefore, $0\preceq\Psi^3_{\delta_\alpha}+\Omega^3_{\delta_\alpha}+\Lambda^3_{\delta_\alpha}\preceq2$.

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Algorithm 1 Multi-Criteria Decision Making (MCDM) Process

1: Start.
2: Input: To select the best alternative.
3: We employ the decision information given in matrix $(\delta_{\alpha\varepsilon})_{m\times n}$.
4: For each alternatives $\mathbb{A}_\epsilon$, $(\epsilon = 1, 2 \ldots, \lambda)$ construct the spherical Fermatean neutrosophic set $\langle \Psi_\epsilon, \Omega_\epsilon, \Lambda_\epsilon; \varrho_\epsilon \rangle$ where $\langle \Psi_\epsilon, \Omega_\epsilon, \Lambda_\epsilon \rangle$ is the center and ϱ_ϵ is the radius of the spherical Fermatean neutrosophic set from the decision matrix $(\delta_{\alpha\varepsilon})_{m\times n}$.
5: Operate spherical Fermatean neutrosophic Frank aggregation operator $\mathbb{SFNFA}\{\alpha_\imath; \imath = 1, 2, \ldots n\}$ to obtain the overall preference values $\mathbb{A}_\epsilon$ $(\epsilon = 1, 2, \ldots \lambda)$.
6: Calculate the cosine similarity measure $\cos(\mathbb{A}_\epsilon, \mathbb{I})$ $(\epsilon = 1, 2, \ldots \lambda)$, where $\mathbb{I} = (1, 0, 0; 1)$ is the positive ideal sphere.
7: The shortest measure value of $COS(\mathbb{A}_\epsilon, \mathbb{I})$ $(\epsilon = 1, 2, \ldots \lambda)$, is the better alternative $\mathbb{A}_\epsilon$, because it is close to the positive ideal alternative $\mathbb{I}$.
8: Rank the alternatives $\mathbb{A}_\epsilon$, $(\epsilon = 1, 2, \ldots \lambda)$ based on the spherical Fermatean neutrosophic aggregation operator $\mathbb{SFNFA}\{\alpha_\imath; \imath = 1, 2, \ldots n\}$ evaluations and cosine similarity measure $\cos(\mathbb{A}_\epsilon, \mathbb{I})$ $(\epsilon = 1, 2, \ldots \lambda)$.
9: Output : Best alternative.
10: End.

5.1 Numerical Example : Selecting the Best Material for a Structural Engineering Project

The complexity of Multi-Criteria Decision Making (MCDM) increases significantly when multiple decision-makers are involved, each with varying perspectives and criteria. Traditional averaging methods often fail to accurately represent the collective decision, leading to suboptimal choices. This issue is especially prevalent in evaluating the commercialization potential of emerging technologies. To address this, we propose an MCDM approach utilizing Spherical Fermatean Neutrosophic Sets, which effectively aggregate diverse decision-makers' evaluations. This method incorporates the Spherical Fermatean Neutrosophic Frank aggregation operator to handle the degrees of membership, neutral membership and non-membership in a decision matrix, ensuring a comprehensive representation of alternatives.

In a practical application, we demonstrate this approach in selecting the best material for a structural engineering project, considering criteria such as structural performance, cost and environmental impact. Decision-makers, including a structural engineer, cost engineer and environmental engineer, evaluate materials like steel, reinforced concrete, composite materials and timber. The spherical Fermatean neutrosophic sets normalize and aggregate their evaluations and the cosine similarity measure identifies the most suitable material. This approach ensures that the chosen alternative aligns with the project's multifaceted requirements, enhancing decision accuracy and project outcomes.

Process Overview:

- **Define the Alternatives and Criteria:** The team identifies four materials (Steel $\mathbb{S}$, Reinforced Concrete $\mathbb{R}$, Composite Materials $\mathbb{C}$, Timber $\mathbb{T}$) and three criteria (Structural Performance, Cost, Environmental Impact).
- **Gather Evaluations from Decision Makers:** Each decision maker (DM1, DM2, DM3) assesses each material against the criteria. For instance: DM1 (Structural Engineer-$\mathbb{SE}$) evaluates based on strength and durability. DM2 (Cost Engineer-$\mathbb{CE}$) evaluates based on initial and long-term costs. DM3 (Environmental Engineer-$\mathbb{EE}$) evaluates based on the material's environmental footprint.
- **Normalize Evaluations:** The scores from each decision maker are converted to geometrical structure as sphere to ensure consistency.
- **Weighting of Criteria:** The team assigns weights to each criterion based on their importance: Structural Performance (0.35), Cost (0.25) and Environmental Impact (0.4).
- **Aggregation of Evaluations using Spherical Fermatean Neutrosophic Frank Aggregation Operator:** The normalized evaluations from all decision makers are aggregated using the $\mathbb{SFNFA}\{\alpha_\imath; \imath = 1, 2, \ldots n\}$ operator to combine their inputs effectively.

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- **Compute the cosine distance for Each Alternative:** The aggregated scores for each criterion are calculate the cosine distance $\cos(\delta_1, \delta_2) =$

$$1 - \frac{\Psi_{\delta_1} \cdot \Psi_{\delta_2} + \Omega_{\delta_1} \cdot \Omega_{\delta_2} + \Lambda_{\delta_1} \cdot \Lambda_{\delta_2}}{\|\Psi_{\delta_1}\| \cdot \|\Psi_{\delta_2}\| + \|\Omega_{\delta_1}\| \cdot \|\Omega_{\delta_2}\| + \|\Lambda_{\delta_1}\| \cdot \|\Lambda_{\delta_2}\|} \times \frac{|\varrho_{\delta_1} - \varrho_{\delta_2}|}{\max(\varrho_{\delta_1}, \varrho_{\delta_2})}$$

  where $\delta_2 = \mathbb{I} = (1, 0, 0; 1)$ is the ideal sphere.

- **Rank the Alternatives:** The materials are ranked based on their overall scores. The shortest distance value of $\cos(\delta_1, \mathbb{I})$ is the better alternative $\mathbb{A}_\epsilon$, because it is close to the ideal alternative $\mathbb{I}$.

- **Select the Best Alternative:** The team selects the material that ranks highest, ensuring it meets the project's structural, financial and environmental requirements.

Alternatives:

1. **Steel($\mathbb{S}$):** Widely used in bridge construction due to its high strength and durability. Can be expensive and requires regular maintenance to prevent corrosion.
2. **Reinforced Concrete ($\mathbb{R}$):** Strong and durable, with good resistance to environmental factors. Typically less expensive than steel but can be heavy and require substantial formwork.
3. **Composite Materials ($\mathbb{C}$):** Made from a combination of materials (e.g., fibers and resin) that offer high strength-to-weight ratios. Generally more expensive but can reduce overall construction time and maintenance.
4. **Timber ($\mathbb{T}$):** Sustainable and renewable material with a lower environmental impact. Generally less durable and may require treatments for increased longevity.

Criteria:

1. **Structural Performance** Assesses the material's strength, durability and ability to withstand loads and environmental conditions over time. Important to ensure the safety and longevity of the bridge.
2. **Cost** Includes initial material costs, transportation, installation and long-term maintenance expenses. Critical to manage the overall budget of the project and ensure cost-effectiveness.
3. **Environmental Impact** Evaluates the ecological footprint of the material, including energy consumption during production, recyclability and environmental degradation over time. Significant for ensuring the sustainability of the project and compliance with environmental regulations.

Step 1: The four possible choices are to be evaluated under these three requirements. The Table 2 presents the Fermatean neutrosophic linguistic evaluation parameters and their corresponding Fermatean neutrosophic values. In Table 3 the Fermatean neutrosophic values are normalized as per beneficiary and non beneficiary criteria.

Step 2: From Fermatean neutrosophic decision matrix in Table 3, we calculate the center, radius using Equation (1), (2) and we frame spherical Fermatean neutrosophic set in Table 4

Step 3: By using the spherical Fermatean neutrosophic Frank aggregate operator we calculated $\mathbb{SFNFA}_{\bigcirc}$ values in Table 5.

Step 4: In Table 6 the aggregated scores for each criterion are calculate the cosine distance $\cos(\delta_1, \delta_2) =$ where $\delta_2 = \mathbb{I} = (1, 0, 0; 1)$ is the ideal sphere.

Step 5 : In Table 7 the materials are ranked based on their overall scores. The shortest distance value is the better alternative because it is close to the ideal alternative $\mathbb{I}$.

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Received: November 26, 2023 Revised: February 15 Accepted: May 27, 2024

Category	**Symbolic representation**	Ψ	Ω	Λ
Very Low	$\mathbb{Z}1$	0.2	0.1	0.5
Low	$\mathbb{Z}2$	0.4	0.3	0.6
Below Average	$\mathbb{Z}3$	0.5	0.4	0.3
Slightly Below Average	$\mathbb{Z}4$	0.6	0.5	0.4
Average	$\mathbb{Z}5$	0.7	0.6	0.3
Slightly Above Average	$\mathbb{Z}6$	0.6	0.5	0.4
Above Average	$\mathbb{Z}7$	0.7	0.3	0.5
High	$\mathbb{Z}8$	0.85	0.2	0.3
Very High	$\mathbb{Z}9$	0.8	0.1	0.4
Extremely High	$\mathbb{Z}10$	0.95	0.05	0.3

Table 1: Fermatean neutrosophic linguistic categories and their corresponding values

DM'S	Materials	$\mathbb{SP}$	$\mathbb{C}$	$\mathbb{EI}$	DM'S	$\mathbb{SP}$	$\mathbb{C}$	$\mathbb{EI}$	DM'S	$\mathbb{SP}$	$\mathbb{C}$	$\mathbb{EI}$
	$\mathbb{S}$	$\mathbb{Z}8$	$\mathbb{Z}7$	$\mathbb{Z}6$		$\mathbb{Z}9$	$\mathbb{Z}7$	$\mathbb{Z}8$		$\mathbb{Z}10$	$\mathbb{Z}7$	$\mathbb{Z}9$
	$\mathbb{R}$	$\mathbb{Z}7$	$\mathbb{Z}9$	$\mathbb{Z}9$		$\mathbb{Z}10$	$\mathbb{Z}8$	$\mathbb{Z}7$		$\mathbb{Z}8$	$\mathbb{Z}6$	$\mathbb{Z}5$
$\mathbb{SE}$	$\mathbb{C}$	$\mathbb{Z}8$	$\mathbb{Z}10$	$\mathbb{Z}10$	$\mathbb{CE}$	$\mathbb{Z}9$	$\mathbb{Z}10$	$\mathbb{Z}8$	$\mathbb{EE}$	$\mathbb{Z}10$	$\mathbb{Z}9$	$\mathbb{Z}8$
	$\mathbb{T}$	$\mathbb{Z}6$	$\mathbb{Z}8$	$\mathbb{Z}7$		$\mathbb{Z}5$	$\mathbb{Z}7$	$\mathbb{Z}6$		$\mathbb{Z}5$	$\mathbb{Z}7$	$\mathbb{Z}6$

Table 2: The Fermatean Neutrosophic Decision Matrix evaluated by three DM's

DM'S	Materials	$\mathbb{SP}$	$\mathbb{C}$	$\mathbb{EI}$
	$\mathbb{S}$	(0.85, 0.2, 0.3)	(0.3, 0.7, 0.5)	(0.6, 0.5, 0.4)
	$\mathbb{R}$	(0.7, 0.3, 0.5)	(0.1, 0.8, 0.4)	(0.8, 0.1, 0.4)
$\mathbb{SE}$	$\mathbb{C}$	(0.85, 0.2, 0.3)	(0.05, 0.95, 0.3)	(0.95, 0.05, 0.3)
	$\mathbb{T}$	(0.6, 0.5, 0.4)	(0.2, 0.85, 0.3)	(0.7, 0.3, 0.5)
	$\mathbb{S}$	(0.8, 0.1, 0.4)	(0.6, 0.7, 0.3)	(0.85, 0.2, 0.3)
	$\mathbb{R}$	(0.95, 0.05, 0.3)	(0.2, 0.85 0.3)	(0.7, 0.3, 0.5)
$\mathbb{CE}$	$\mathbb{C}$	(0.8, 0.1, 0.4)	(0.05, 0.95, 0.3)	(0.85, 0.2, 0.3)
	$\mathbb{T}$	(0.7, 0.6, 0.3)	(0.7, 0.3, 0.5)	(0.6, 0.5, 0.4)
	$\mathbb{S}$	(0.95, 0.05, 0.3)	(0.3, 0.7, 0.5)	(0.8, 0.1, 0.4)
$\mathbb{EE}$	$\mathbb{R}$	(0.85, 0.2, 0.3)	(0.5, 0.6, 0.4)	(0.7, 0.6, 0.3)
	$\mathbb{C}$	(0.95, 0.05, 0.3)	(0.1, 0.8, 0.4)	(0.85, 0.2, 0.3)
	$\mathbb{T}$	(0.7, 0.6, 0.3)	(0.3, 0.7, 0.5)	(0.6, 0.5, 0.4)

Table 3: The Fermatean Neutrosophic Normalized Decision Matrix

Materials	$\mathbb{SP}$	$\mathbb{C}$	$\mathbb{EI}$
$\mathbb{S}$	(0.87, 0.12, 0.33; 0.11)	(0.40, 0.70, 0.43; 0.24)	(0.75, 0.27, 0.37; 0.28)
$\mathbb{R}$	(0.83, 0.18, 0.37; 0.22)	(0.27, 0.75, 0.37; 0.28)	(0.73, 0.33, 0.40; 0.29)
$\mathbb{C}$	(0.87, 0.12, 0.33; 0.11)	(0.07, 0.90, 0.33; 0.12)	(0.88, 0.15, 0.30; 0.11)
$\mathbb{T}$	(0.67, 0.57, 0.33; 0.12)	(0.27, 0.75, 0.43; 0.18)	(0.63, 0.43, 0.43; 0.16)

Table 4: The Spherical Fermatean Neutrosophic Decision Matrix

Materials	$\mathbb{SFNFA}_{\bigcirc=2}$	$\mathbb{SFNFA}_{\bigcirc=3}$
$\mathbb{S}$	(0.171, 0.017, 0.051; 0.004)	(0.394, 0.017, 0.051; 0.261)
$\mathbb{R}$	(0.147, 0.037, 0.055; 0.019)	(0.376, 0.038, 0.055; 0.019)
$\mathbb{C}$	(0.225, 0.011, 0.033; 0.002)	(0.435, 0.011, 0.033; 0.002)
$\mathbb{T}$	(0.076, 0.165, 0.062; 0.004)	(0.320, 0.166, 0.062; 0.004)
Materials	$\mathbb{SFNFA}_{\bigcirc=4}$	$\mathbb{SFNFA}_{\bigcirc=5}$
$\mathbb{S}$	(0.480, 0.017, 0.051; 0.359)	(0.528, 0.018, 0.051; 0.413)
$\mathbb{R}$	(0.464, 0.039, 0.055; 0.019)	(0.512, 0.039, 0.055; 0.019)
$\mathbb{C}$	(0.517, 0.011, 0.033; 0.002)	(0.562, 0.012, 0.033; 0.002)
$\mathbb{T}$	(0.414, 0.167, 0.062; 0.004)	(0.465, 0.168, 0.062; 0.004)

Table 5: $\mathbb{SFNFA}$ Aggregated Values

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Materials	COS($\mathbb{S}$, $\mathbb{I}$)	COS($\mathbb{R}$, $\mathbb{I}$)	COS($\mathbb{C}$, $\mathbb{I}$)	COS($\mathbb{T}$, $\mathbb{I}$)
$\mathbb{SFNFA}_{\bigcirc=2}$	0.0494	0.1057	0.0131	0.6051
$\mathbb{SFNFA}_{\bigcirc=3}$	0.2681	0.0338	0.0048	0.1286
$\mathbb{SFNFA}_{\bigcirc=4}$	0.3632	0.0288	0.0039	0.0850
$\mathbb{SFNFA}_{\bigcirc=5}$	0.4156	0.0270	0.0036	0.0701

Table 6: Cosine Distance Measure Between $\mathbb{I}$ and between $\mathbb{SFNFA}$ Aggregated Values

Method	Ranking	Best Materials
$\mathbb{SFNFA}_{\bigcirc=2}$	$\mathbb{C} < \mathbb{S} < \mathbb{R} < \mathbb{T}$	$\mathbb{C}$
$\mathbb{SFNFA}_{\bigcirc=3}$	$\mathbb{C} < \mathbb{R} < \mathbb{T} < \mathbb{S}$	$\mathbb{C}$
$\mathbb{SFNFA}_{\bigcirc=4}$	$\mathbb{C} < \mathbb{R} < \mathbb{T} < \mathbb{S}$	$\mathbb{C}$
$\mathbb{SFNFA}_{\bigcirc=5}$	$\mathbb{C} < \mathbb{R} < \mathbb{T} < \mathbb{S}$	$\mathbb{C}$

Table 7: The results of the sensitivity analysis.

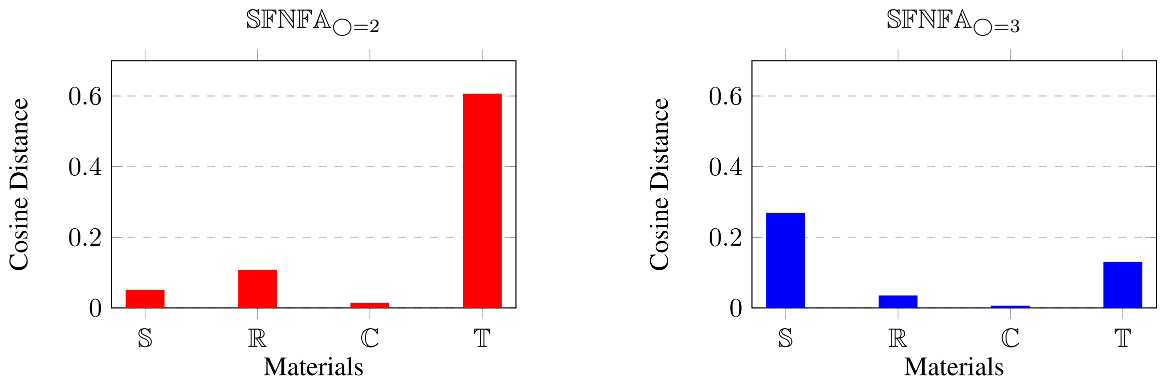

Figure 2: Comparison of $\mathbb{SFNFA}_{\bigcirc=2}$ and $\mathbb{SFNFA}_{\bigcirc=3}$ for different materials

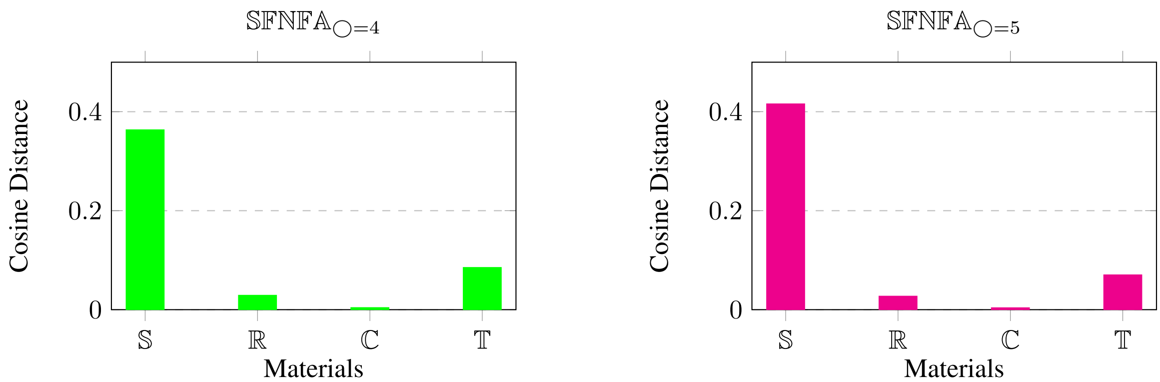

Figure 3: Comparison of $\mathbb{SFNFA}_{\bigcirc=4}$ and $\mathbb{SFNFA}_{\bigcirc=5}$ for different materials

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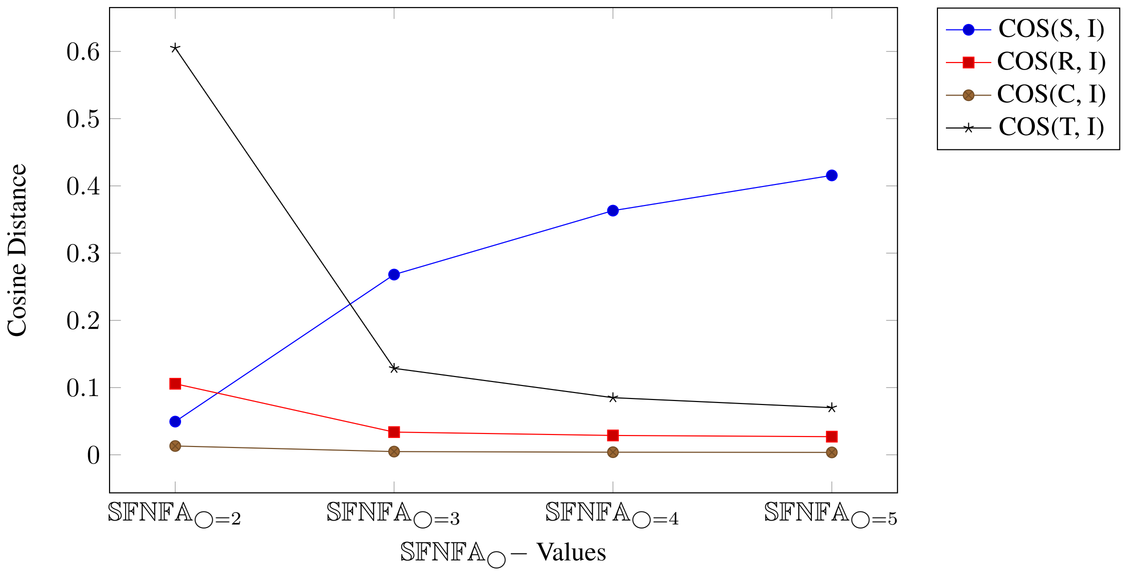

Figure 4 :Comparison of $\mathbb{SFNFA}_{\bigcirc=2,3,4,5}$ for different materials

5.2 Comparative Analysis

We compare our proposed $\mathbb{SFNFA}$ method with a existing cubic spherical neutrosophic aggregation methods $\mathbb{CSNWA}^{A}_{\wp}$ $\mathbb{CSNWA}^{A}_{\rho}$ $\mathbb{CSNWG}^{A}_{\wp}$ $\mathbb{CSNWG}^{A}_{\rho}$ proposed by Krishnaprakash et al.[13] and $\mathbb{CSNWAAO}$, $\mathbb{CSNWGAO}$ by Gomathi et al.[9]

Method	Ranking	Best Materials
$\mathbb{CSNWA}^{A}_{\wp}$ [13]	$\mathbb{C}<\mathbb{R}<\mathbb{T}<\mathbb{S}$	$\mathbb{C}$
$\mathbb{CSNWA}^{A}_{\rho}$ [13]	$\mathbb{C}<\mathbb{R}<\mathbb{T}<\mathbb{S}$	$\mathbb{C}$
$\mathbb{CSNWG}^{A}_{\wp}$ [13]	$\mathbb{C}<\mathbb{S}<\mathbb{R}<\mathbb{T}$	$\mathbb{C}$
$\mathbb{CSNWG}^{A}_{\rho}$ [13]	$\mathbb{C}<\mathbb{R}<\mathbb{T}<\mathbb{S}$	$\mathbb{C}$
$\mathbb{CSNWAAO}$ [9]	$\mathbb{C}<\mathbb{R}<\mathbb{T}<\mathbb{S}$	$\mathbb{C}$
$\mathbb{CSNWGAO}$ [9]	$\mathbb{C}<\mathbb{T}<\mathbb{R}<\mathbb{S}$	$\mathbb{C}$
$\mathbb{SFNFA}_{\bigcirc=2}$	$\mathbb{C}<\mathbb{S}<\mathbb{R}<\mathbb{T}$	$\mathbb{C}$
$\mathbb{SFNFA}_{\bigcirc=3}$	$\mathbb{C}<\mathbb{R}<\mathbb{T}<\mathbb{S}$	$\mathbb{C}$
$\mathbb{SFNFA}_{\bigcirc=4}$	$\mathbb{C}<\mathbb{R}<\mathbb{T}<\mathbb{S}$	$\mathbb{C}$
$\mathbb{SFNFA}_{\bigcirc=5}$	$\mathbb{C}<\mathbb{R}<\mathbb{T}<\mathbb{S}$	$\mathbb{C}$

Table 8: The results of the sensitivity analysis.

The consistency in material rankings across different methods suggests robustness in the evaluation criteria used, particularly favoring composite materials ($\mathbb{C}$) as optimal for various applications. Proposed $\mathbb{SFNFA}$ method's performance aligns well with these established approaches, reinforcing its validity and reliability in material selection processes.

Overall, the comparative analysis indicates that the $\mathbb{SFNFA}$ method offers competitive performance in material selection, demonstrating a reliable framework for decision making comparable to existing state-of-the-art methods in cubic spherical neutrosophic aggregation. This consistency across methods strengthens confidence in the suitability of composite materials as preferred choices across different contexts and configurations.

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6 Conclusion and Future Work

The introduction of the Spherical Fermatean Neutrosophic Set (SFNS) represents a significant advancement in the realm of neutrosophic sets and Fermatean neutrosophic sets. This study addressed the limitations of traditional decision making approaches, particularly in contexts with diverse perspectives, by proposing SFNS as a comprehensive solution. By encapsulating membership, non-membership and indeterminacy functions within a spherical representation, SFNS offers a holistic perspective that better accommodates the complexities inherent in decision making processes.

Through empirical testing and analysis, it was demonstrated that SFNS excels in resolving ambiguity and risk with greater accuracy and effectiveness compared to conventional methods. This innovative approach provides decision-makers with a nuanced tool to navigate complex scenarios involving diverse perspectives.

Moving forward, further research and application of SFNS in real-world decision making contexts will be crucial to fully realize its potential benefits. Future studies should focus on refining the SFNS model, exploring its application across various industries and comparing its performance with other advanced decision making frameworks. By continuing to explore and refine SFNS, we can enhance decision making processes and contribute to more informed and resilient decision outcomes in diverse and dynamic environments.

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